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Heat diffusion as a source of aerodynamic sound

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The paper examines the role of heat diffusion as an internal noise source in aero-engines and as a source of noise in the mixing of hot jets. We consider a number of model problems and find that the sound induced by unsteady heat transfer can show an unusually weak dependence on the mean flow velocity U, the intensity scaling as U^3 in three dimensions. At low enough velocities diffusion effects will overwhelm other noise sources, but we have failed in our search for a significant practical situation in which we can prove that sound generated by diffusion clearly dominates over that excited by unsteady aerodynamic forces; they are sometimes comparable.

We examine the possibility that diffusive monopole sources feature in the noise of hot jets using model problems in the linear case and using dimensional analysis in the nonlinear case, and conclude that no significant monopole exists when the specific heats are constant. But they are not constant at low frequencies when, for example, heat flows into and out of vibrational energy modes; then an important monopole source is present. This source shows an unusually complicated scale effect.

1. Introduction

T'' e excess noise of aeroengines above that predicted by Lighthill's (1952, 1954) theory of aerodynamic sound generation at low Mach numbers has led many investigators to search for new source mechanisms, or alternatively for scattering mechanisms that can enhance the radiation efficiency of Lighthill's quadrupoles (Ffowcs Williams & Hall 1970; Crighton 1972; Morfey 1973, and others). All searched for mechanisms that generate noise that increases with velocity less rapidly than the quadrupole noise, and which might dominate at low jet speeds. But in this search little attention has been paid until very recently to the roles of heat diffusion and viscous dissipation. These are usually thought to be irrelevant to the noise problem since the Reynolds number involved is so large.

Even though the Reynolds number based on the mean velocity and nozzle scale is extremely large, turbulence ensures that the appropriate length scale is small enough for irreversible processes rapidly to dissipate the energy of a jet. So when examining engine noise it might seem premature to disregard diffusion out of hand on the grounds that it is too slow over engine scales. The rapid oscillations of Trevelyan's rocker are possible since "the conduction of heat is not a slow process when small distances and masses are in question" (Rayleigh 1877).

Inhomogeneities in temperature can produce internal noise by scattering hydrodynamic pressures (Cumpsty & Marble 1974; Ffowcs Williams & Howe 1974), but the role of heat diffusion as an internal noise source has been examined in detail only for combustion noise and in certain resonance situations: the Rijke tube, the Bosscha tube and tubes that are hotter at their closed end than at their open end (Rayleigh 1877; Rott 1969). Temperature differences are known to have an effect on jet noise. It has been observed for model jets (Hoch *et al.* 1973) that an increase in the jet temperature reduces the noise at high jet velocities but increases the noise at low jet velocities; this increase was not attributed by Hoch to rig noise but was considered an essential feature of heated jets. Explanations of these effects have been provided by theoretical work by Morfey (1973, 1974), Mani (1974), Tester & Morfey (1976) and others. They suggest that temperature inhomogeneities can scatter efficiently the quadrupoles' near field, and can change the amount of mean profile refraction and shrouding of the quadrupole noise. Lush & Fisher (1973), however, have formed an empirical correlation of the hot-jet noise data by assuming the existence of a source that generates noise whose intensity scales with the fourth power of velocity. Crighton (1975) argues the rationality of this and concludes that irreversible processes induce such a monopole; we disagree with his conclusion.

In this paper we begin our study by considering certain model problems where the temperature is specified on some solid bodies. From these we confirm that the sound may be considered to arise from the monopole-like fluctuating heat transfer to the fluid. We find that the sound intensity sometimes scales with the cube of the mean velocity, an unusually low exponent for noise in three dimensions. Such low exponents are possible since an additional length scale is present owing to diffusion, but are achieved at the expense of introducing inverse powers of the numerically large Péclet number $Pe = U^2/\omega\kappa$, the 'Reynolds' number based on the diffusivity of heat rather than the kinematic viscosity; the intensity of the sound scales as M^4Pe^{-1}. Thus, when in § 4 the heat transfer is estimated for certain aeroengine and underwater situations, the resulting radiated sound is shown to be often insignificant.

The case when streams of different temperatures mix and there is no fluctuating heat input into the fluid is dealt with in § 5. In the linear theory the expansion as part of the fluid is heated is compensated for by the contraction as another part is cooled. The noise source degenerates to a dipole, and the intensity of the radiated sound then scales with the mean velocity to the fourth power. Again, however, this low exponent involves the introduction of inverse powers of the Péclet number and is too weak to provide a theoretical justification of the correlation obtained by Lush & Fisher (1973); in any event this result is restricted to laminar flows.

In § 6 Crighton's arguments for the existence of diffusive monopoles in turbulent flows are discussed and shown to be in error. We show that, even in the nonlinear theory, no monopole source exists that is capable of generating an acoustic field whose intensity scales as U^4, provided only that the specific heat remains constant. In § 7 we examine the effect of variations in the specific heats. With such variations heat diffusion does induce a monopole source that generates a low frequency far field with intensity scaling as U^4 and with no dependence on Péclet number. However this source is much weaker than the work of Crighton would suggest.

2. Sound generated by heat diffusing from solid bodies

Landau & Lifshitz (1959, p. 287) have examined the sound radiated when the temperature of a solid body is made to oscillate. The fluid close to the body expands and contracts as heat is fed in and is extracted, and the consequent flow of mass acts like a monopole source of sound. For a plane surface whose temperature varies slowly they deduce an inner expansion for the temperature, valid in a region close to the surface where the diffusion equation holds. The resulting velocity of the fluid normal to the surface attains a limiting value at distances from the surface small compared with an acoustic wavelength, and provides the matching condition to determine the radiated sound.

It is possible alternatively to use an acoustic analogy. The analogies of Lighthill (1952, 1954), Phillips (1960) and Howe (1975) all reduce to the form

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\rho_0}{T_0}\frac{\partial^2 T}{\partial t^2}$$

for the linear problem, and the radiated sound may be deduced if the inner expansion is used to determine $\partial^2 T/\partial t^2$ (see Howe 1975). Here p is the pressure, ρ the density, T the temperature, c the ambient speed of sound and a suffix zero indicates that the undisturbed value of the variable is to be taken.

A further alternative is to use the 'inner solution' to deduce the fluctuating heat flow from the body to the fluid; the heat flow from a surface element $\mathbf{dS}$ is $-k\nabla T.\mathbf{dS}$, where k is the thermal conductivity. We deduce the radiated sound from the equation (Morse & Ingard 1968, equations 7.1.21 and 7.1.22)

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{1}{c_p T_0}\frac{\partial Q}{\partial t} - \nabla . \mathbf{F}, \tag{1}$$

where c_p is the specific heat at constant pressure, Q is the rate of heat addition and $\mathbf{F}$ is the applied force, both per unit volume. For this problem $\mathbf{F}$ is zero.

These three methods yield the same result. The monopole source strength is given by $\int \rho_0 \mathbf{u}.\mathbf{dS}$ if the first method is used, and the velocity $\mathbf{u}$ is matched by

$$\int \frac{\rho_0}{T_0}\frac{\partial T}{\partial t}dV$$

if an acoustic-analogy approach is adopted and by

$$\int \frac{k}{c_p T_0}\nabla T.\mathbf{dS}$$

if the heat flow is calculated. Volume and surface integrals are to be evaluated over the source region and the surface bounding it. These source terms are equivalent in the linear theory if pressure variations may be neglected in the source region, i.e. the source is compact. However, because mistakes have been made when applying the method of matched expansions loosely in this manner, it is wise to verify directly for this simple problem that the results obtained by the above approaches are correct.

The governing equations for a perfect gas are

$$p = R\rho T, \tag{2}$$

$$\partial\rho/\partial t + \partial(\rho u_i)/\partial x_i = 0, \tag{3}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{4}$$

$$\rho c_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right), \tag{5}$$

where
$$\tau_{ij} = \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) + (\mu_v - \tfrac{2}{3}\mu)\,\delta_{ij}\,\partial u_k/\partial x_k.$$

(See, for example, Lighthill 1956.) μ is the viscosity, μ_v the bulk viscosity and R the specific gas constant, which is equal to $c_p - c_v$, where c_v is the specific heat at constant volume. These equations can be combined and linearized to give the governing equation for small variations in temperature as

$$\frac{\partial^3 T}{\partial t^3} - c^2\nabla^2\frac{\partial T}{\partial t} + c^2\kappa\nabla^4 T - (\tfrac{4}{3}\nu + \gamma\kappa)\,\nabla^2\frac{\partial T}{\partial t} + \tfrac{4}{3}\nu\gamma\kappa\nabla^4\frac{\partial T}{\partial t} = 0, \tag{6}$$

where $\gamma = c_p/c_v$ is the ratio of the specific heats, $\kappa = k/\rho_0 c_p$ is the diffusivity of heat, and $\nu = \mu/\rho_0$ is the kinematic viscosity ($\nu = (\mu + \tfrac{3}{4}\mu_v)/\rho_0$ if we include the effects of bulk viscosity). For periodic disturbances at low frequencies

$$(\omega \ll c^2/\kappa \quad \text{and} \quad \omega \ll c^2/\nu)$$

there are two linearly independent solutions to (6). These are, in one space dimension,

$$T \sim \exp\{-i\omega t + i\omega x[1 + \tfrac{1}{2}i\omega(\tfrac{4}{3}\nu + (\gamma-1)\kappa)/c^2 + \ldots]/c\}, \tag{7}$$

the solution to the wave equation modified by slight damping, and

$$T \sim \exp\{-i\omega t - (1-i)\,x(\tfrac{1}{2}\omega/\kappa)^{\frac{1}{2}}[1 - \tfrac{1}{2}i\omega(\gamma-1)\,(\kappa - \tfrac{4}{3}\nu)/c^2 + \ldots]\}, \tag{8}$$

the solution to the diffusion equation modified by slight compressibility. The boundary conditions, i.e. the specified temperature and the condition of vanishing velocity, determine the magnitude of the acoustic and diffusion waves; if the rigid surface at $x = 0$ has temperature $T_0 + T_1\,e^{-i\omega t}$, to first order in T_1/T_0 and lowest order in $\omega\kappa/c^2$ the radiated sound has fluctuating pressure

$$p = \rho_0 c^2 \frac{T_1}{T_0}\left(\frac{\omega\kappa}{c^2}\right)^{\frac{1}{2}} \exp\{-i\omega(t - x/c) - \tfrac{1}{4}i\pi\}. \tag{9}$$

Since the three methods using the inner expansion give results in agreement with this direct approach, we feel justified in using them in more complicated situations examined later that are not amenable to a direct approach.

These three methods and the direct approach have also been used to deduce the sound radiated when the temperature of a sphere of radius a is made to vary as $T_0 + T_1 e^{-i\omega t}$ in the limit $\omega a/c \ll 1$ but $(\omega/\kappa)^{\frac{1}{2}}a \gg 1$ (the sphere is small on the wavelength scale but large when compared with the width of the thermal boundary layer). All the approaches agree and give the far-field fluctuating pressure as

$$p \approx \rho_0 c^2 \frac{T_1}{T_0}\left(\frac{\omega\kappa}{c^2}\right)^{\frac{1}{2}} \frac{\omega a}{c}\frac{a}{r}\exp\{-i\omega(t-r/c)-\tfrac{3}{4}i\pi\}. \tag{10}$$

In a similar manner we can demonstrate that heat transfer causes a reduction in amplitude on reflexion when a plane sound wave is normally incident on a hard constant-temperature surface at $x = 0$. The sound wave causes the temperature of the fluid to vary as

$$T = T_0 + T^* \exp\{-i\omega t - i\omega x[1+\tfrac{1}{2}i\omega(\tfrac{4}{3}\nu+(\gamma-1)\kappa)/c^2+\ldots]/c\}$$

[cf. (7)]. The requirement of zero normal velocity ensures that the wave is reflected without change in amplitude to lowest order in $\omega\kappa/c^2$. But the condition that the temperature of the wall be fixed at T_0 means that a thermal wave must also be present; this is given, to lowest order in $\omega\kappa/c^2$, by

$$-2T^* \exp\{-i\omega t + (1-i)\,x(\tfrac{1}{2}\omega/\kappa)^{\frac{1}{2}}\}$$

and drives an acoustic wave with temperature

$$-2^{\frac{1}{2}}(1-i)\,T^*(\gamma-1)\,(\omega\kappa/c^2)^{\frac{1}{2}}\exp\{-i\omega t+i\omega x\}.$$

So we deduce that the amplitude of the reflected wave is smaller than that of the incident wave by the factor

$$1-2^{\frac{1}{2}}(\gamma-1)\,(\omega\kappa/c^2)^{\frac{1}{2}}+O(\omega\kappa/c^2)$$

(cf. Cremer 1948). Thus a hard constant-temperature surface absorbs energy from incident sound waves, regardless of the surface's absolute temperature.

3. Sound generated by fluctuating flows over hot bodies

An alternative means exists whereby the heat transfer from a hot body to the surrounding fluid may vary and cause sound to be generated. If the fluid velocity over the body fluctuates, or the rigid body itself undergoes slight oscillations in position, the heat transfer from the body is unsteady and produces monopole-type noise.

To highlight the role of heat diffusion we restrict attention to inviscid flows. Not only does this simplify the governing equations, but it also enables us to pose problems that would otherwise not be sensible (for example, if we must include the viscous boundary layer, we cannot discuss flow over an infinite flat plate). Thus, as a model problem in this class, we can consider the two-dimensional laminar flow over the hard wall $\xi_2 = 0$ of an inviscid fluid with unperturbed temperature T_0 and with velocity U in the ξ_1 direction. The wall temperature is specified as T_0 for $\xi_1 < 0$ and for $\xi_1 > a$ but is T_1 for $0 < \xi_1 < a$. The wall is then allowed to execute small rigid oscillations with velocity $-U^*e^{-i\omega t}$ in the ξ_1 direction.

Variations in temperature due to diffusion will occur only in regions close to the temperature discontinuities, within a distance of the order of the diffusion length

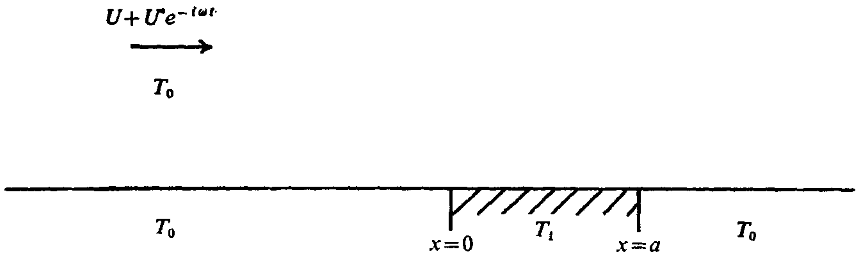

FIGURE 1. Flow over a partly heated wall.

scale $(\kappa/\omega)^{\frac{1}{2}}$. At low enough frequencies, $\omega\kappa/c^2 \ll 1$, the source regions will be compact on a wavelength scale, so we neglect pressure variations within them. Furthermore we consider the wall to be stationary and apply locally a Galilean transformation such that the fluid velocity varies as $U+U^*e^{-i\omega t}$ (figure 1). These assumptions introduce a relative error $O(M^2)$.

The temperature equation (5) becomes

$$\rho c_p DT/Dt = \nabla.(k\nabla T) \tag{11}$$

in the frame of reference $(x_1, x_2, x_3) \equiv (x, y, z)$ in which the wall is stationary. For the linear problem, the steady temperature satisfies $T = T_0 + T'$, where

$$U\partial T'/\partial x = \kappa\nabla^2 T', \tag{12}$$

with $$T' = (T_1 - T_0)[H(x) - H(x-a)] \quad \text{on} \quad y = 0.$$

We treat the fluctuating velocity $U^*e^{-i\omega t}$ as a small perturbation to the steady velocity U. Then the fluctuating part of the temperature satisfies to first order in U^*

$$\partial T^*/\partial t + U\partial T^*/\partial x - \kappa\nabla^2 T^* = -U^*e^{-i\omega t}\partial T'/\partial x, \tag{13}$$

with $$T^* = 0 \quad \text{on} \quad y = 0.$$

We can thus find the 'inner solution', an approximation for the temperature in the source region.

To deduce the radiated sound, to match on an acoustic wave, we employ Lighthill's acoustic analogy in the frame of reference $\boldsymbol{\xi}$ in which the fluid at infinity is at rest; the frames of reference $\boldsymbol{\xi}$ and $\mathbf{x}$ are related by

$$\xi_1 = x_1 - Ut - iU^*(e^{-i\omega t} - 1)/\omega, \quad \xi_2 = x_2, \quad \xi_3 = x_3,$$

and the Jacobian of the transformation is unity, i.e. $dV_{\xi} = dV_{\mathbf{x}}$. We have (Lighthill 1952)

$$\partial^2\rho/\partial t^2 - c^2\nabla^2\rho = \partial^2 T_{ij}/\partial x_i\,\partial x_j,$$

where $$T_{ij} = \rho u_i u_j + (p - c^2\rho)\,\delta_{ij}.$$

In the manner of Lighthill (1952), but taking account of the variation of M_r with time, we deduce that

$$(\rho-\rho_0)(\boldsymbol{\xi}, t) = \frac{\xi_i\xi_j}{4\pi c^4|\boldsymbol{\xi}|^3}\iiiint \frac{\partial}{\partial\tau}\left\{\frac{1}{(1-M_r)}\frac{\partial}{\partial\tau}\left[\frac{T_{ij}(\mathbf{x}_0,\tau)}{(1-M_r)}\right]\right\}\delta(\tau - t + r/c)\,dV_{\mathbf{x}_0}\,d\tau, \tag{14}$$

where $$M_r = -(U/c + U^*e^{-i\omega\tau}/c)\,\xi_1/|\boldsymbol{\xi}|$$

and $$r = |\boldsymbol{\xi} - \boldsymbol{\xi}_0(\mathbf{x}_0, \tau)|.$$

A similar expression has been obtained by Ffowcs Williams (1974), who uses Lagrangian co-ordinates. His result differs since the Jacobian of the transformation is then a density ratio. In our problem

$$T_{ij} = -c^2(\rho-\rho_0)\,\delta_{ij} = c^2\rho_0\,(T-T_0)\,\delta_{ij}/T_0,$$

where T is the 'inner solution' for the temperature. We are thus able to deduce the radiated sound (see appendix A).

If $\omega a/c \ll 1$, i.e. the heated region of the wall is compact on a wavelength scale, the fluctuating pressure of the radiated sound at $\xi_1 = R\cos\theta$, $\xi_2 = R\sin\theta$ is

$$p = \frac{\rho_0 c^2}{(1+M\cos\theta)^2}\,\frac{T_1-T_0}{T_0}\left(\frac{\tilde{\omega}\kappa}{c^2}\right)^{\frac{1}{2}}\frac{U^*}{c}\,\frac{\tilde{\omega}a\cos\theta}{c}\,\frac{\exp\{-i\tilde{\omega}(t-R/c)+\frac{1}{4}i\pi\}}{(2\pi\tilde{\omega}R/c)^{\frac{1}{2}}}, \tag{15}$$

with a relative error $O(\omega a/c)+O(M^2)+O(U^*/c)+O(\omega\kappa/c^2)+O((T_1-T_0)/T_0)$, where $M = U/c$ and $\tilde{\omega}$ is the Doppler-shifted frequency $\omega/(1+M\cos\theta)$. The noise is dipole induced since the heat input near $x = 0$ is compensated for by the heat extraction near $x = a$.

In the limit $a\to\infty$, corresponding to flow over a rigid oscillating wall with one step discontinuity in temperature, the artificial dipole nature of the source is removed and the far-field fluctuating pressure is

$$p = \frac{\rho_0 c^2}{(1+M\cos\theta)^2}\,\frac{T_1-T_0}{T_0}\left(\frac{\tilde{\omega}\kappa}{c^2}\right)^{\frac{1}{2}}\frac{U^*}{c}\,\frac{\exp\{-i\tilde{\omega}(t-R/c)\}}{(2\pi\tilde{\omega}R/c)^{\frac{1}{2}}}, \tag{16}$$

with a relative error $O(M^2)+O(U^*/c)+O(\omega\kappa/c^2)+O((T_1-T_0)/T_0)$.

In appendix B we show that, in the limiting case $a\to\infty$, the sound radiated is unaltered if the boundary conditions on $y = 0, x < 0$ are changed from $T = T_0$ to $\partial T/\partial y = 0$. We still specify $T = T_1$ for $y = 0, x > 0$, and then we have the problem of flow over a heated juddering semi-infinite flat plate.

If, as is usually assumed in aerodynamic noise theory, frequencies and all velocities scale with the mean velocity U, the far-field acoustic intensity associated with the heat-input monopole in these unsteady laminar flows scales as U, U^2 and U^3 respectively in one, two and three dimensions (the compact case expressed in (15) differs because the source degenerates to a dipole). The change from the normal U^4 monopole scaling in three dimensions arises because the intensity of the radiated sound depends on $\omega\kappa/U^2$, the reciprocal of the Péclet number, and scales as M^4Pe^{-1}.

4. Practical estimates of the noise from heat sources

In aeroengine and underwater acoustics, there are several situations where heat sources could be important. For example, there will be fluctuating heat transfer when the flow over turbine blades of high heat capacity has varying temperature at the outlets to the combustion chamber, when the flow over cooled turbine blades has varying velocity and when over cooled turbine blades there is unsteady laminar-to-turbulent boundary-layer transition or there are unsteady shocks.

To assess the importance of some of these sources, we compare the noise resulting from the unsteady heat transfer with that resulting from the unsteady viscous boundary-layer drag. We deduce the radiated sound from (1):

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = -\nabla . \mathbf{F} + \frac{1}{c_p T_0}\frac{\partial Q}{\partial t},$$

where $\mathbf{F}$ is the applied force and Q the rate of heat addition. For Prandtl numbers of order unity the effects on incompressible fluid motion of heat diffusion and viscous dissipation are comparable; but the viscous drag is a dipole noise source, and so the sound pressure generated by it will be smaller than that generated by the heat-transfer monopole by a factor of the order of the mean-flow Mach number. However, in most applications a far more dominant source arises from the fluctuations in lift and in form drag. At high Reynolds number these forces are much greater than the skin-friction drag and provide the most important noise source.

We obtain an estimate of the relative importance of these sources in laminar flows by a dimensional argument. We consider unsteady laminar flow of a fluid at temperature T_0 over an aerofoil of length a whose temperature is T_1. From §2 or §3 for the inviscid case, and from the estimates of Chapman (1974) or of Gersten (1965) for viscous flow, the fluctuating heat input scales as $(Ua/\kappa)^{\frac{1}{2}} k(T_1 - T_0)$ per unit length, if frequencies scale as U/a. The fluctuating viscous drag scales as $(Ua/\nu)^{\frac{1}{2}} \mu U$ (Chapman 1974; Gersten 1965) and the fluctuating lift and form drag scale as $\rho U^2 a$. We deduce the resulting noise from (11). The far-field pressure scales as

$$\rho_0 c^2 M^2 \left(\frac{Ua}{\kappa}\right)^{-\frac{1}{2}} \frac{T_1 - T_0}{T_0}, \quad \rho_0 c^2 M^3 \left(\frac{Ua}{\nu}\right)^{-\frac{1}{2}}, \quad \rho_0 c^2 M^3$$

for the heat source, the viscous drag source and the lift or form drag source respectively. For a typical configuration $(Ua/\kappa)^{-\frac{1}{2}} \sim 10^{-3}$ and the noise from the fluctuating lift and form drag will be dominant.†

As a more definite example, we examine flow of very slowly varying density incident on constant-temperature blading. The lift per unit length experienced by a two-dimensional aerofoil at an angle of incidence of about 5° is given approximately by $\frac{1}{4}\rho U^2 a$, where a is the chord length. The ratio of lift to drag is a maximum, with a value of the order of 60, at about this angle of incidence (Goldstein 1965). The boundary-layer drag and heat transfer between a flat plate of length a and the surrounding fluid are given by

$$D \approx 0{\cdot}73 \rho U^2 a\, (Ua/\nu)^{-\frac{1}{2}}$$

and

$$Q \approx 0{\cdot}73\, k \Delta T (Ua/\nu)^{\frac{1}{2}} (\mu c_p/k)^{\frac{1}{3}}$$

for a turbulent boundary layer and by

$$D \approx 1{\cdot}32 \rho U^2 a (Ua/\nu)^{-\frac{1}{2}}$$

and

$$Q \approx 1{\cdot}32\, k \Delta T (Ua/\nu)^{\frac{1}{2}} (\mu c_p/k)^{\frac{1}{3}}$$

† This result will hold at both high and low Strouhal numbers $\omega a/U$ for viscous flows. At low Strouhal number the drag scales as $\rho U^2 a$, while at high Strouhal number the inertial drag dominates, scaling as $\rho U \omega a^2$.

for a laminar boundary layer, where ΔT is the temperature difference between the plate and the fluid (Chapman 1974). Setting $\rho = \rho_0 + \rho' e^{-i\omega t}$ and $\Delta T = T_0 \rho' e^{-i\omega t}/\rho_0$, we obtain estimates for the fluctuating drag and heat transfer, valid in the 'very slowly varying' approximation, for an aerofoil in a flow of varying temperature. For a Prandtl number of order unity, the sound pressure resulting from the oscillating drag on a flat plate is smaller by a factor of the flow Mach number than that resulting from the oscillating heat input. However the sound pressure resulting from the oscillating lift on an aerofoil is larger by factors of about $\frac{1}{3}(Ua/\nu)^{\frac{1}{5}}M$ and $\frac{1}{5}(Ua/\nu)^{\frac{1}{2}}M$ for the cases of turbulent and laminar boundary layers respectively. For a typical engine configuration these factors are about unity and 50 respectively. Thus for laminar boundary layers, except at extremely low Mach numbers, the fluctuating heat input will not be an important noise source, but when the boundary layers are turbulent this heat source is probably significant.

If the flow velocity rather than the flow temperature is allowed to vary, in the laminar case heat sources become less important and the noise produced is as small as that due to the skin-friction drag. Suppose that the fluid velocity over a cooled two-dimensional flat plate of length a varies as $U + U^* e^{-i\omega t}$, where the frequency is so low that $\omega a/c \ll 1$ and the quasi-steady approximation may be used to deduce the drag and heat transfer. For laminar boundary layers the fluctuating drag is $0{\cdot}25\mu U^*(Ua/\nu)^{\frac{1}{2}} e^{-i\omega t}$ and the fluctuating heat transfer is $0{\cdot}075k(T_0 - T_1)(Ua/\nu)^{\frac{1}{2}} e^{-i\omega t} U^*/U$ (Gersten 1965), where T_0 and T_1 are the temperatures of the fluid and plate respectively. The ratio of the sound pressure generated by the viscous drag to that generated by the heat transfer is

$$\tfrac{10}{3}(\mu c_p/k)\, MT_0/(T_0 - T_1) : 1.$$

Typically, in aeroengines we find that these noise sources are comparable. In underwater contexts the Mach numbers involved are smaller, but both the Prandtl number $\mu c_p/k$ and the factor $T_0/(T_0 - T_1)$ are large; then the skin-friction drag dominates.

Thus, although diffusion effects will overwhelm aerodynamic sources at low enough Mach number, we have not been able to think of a significant practical situation in which we can show that sound generated by diffusion clearly dominates that excited by unsteady aerodynamic forces. The two noise sources are comparable, however, when boundary layers are turbulent. But we expect our simple modelling to contain only an order-of-magnitude guide to the far more complicated practical situations, so should not be surprised to find that unsteady heat transfer induces monopoles accounting for significant sources of engine noise.

5. Sound generated by unsteady heat transfer between gas streams

Even when there is no heat transfer from foreign bodies, heat diffusion may still be a source of sound if inhomogeneities exist in the fluid temperature. Then, because the expansion of parts of the fluid being heated is exactly compensated for in the linear theory by the contraction of the neighbouring parts being cooled, there is no longer any monopole source term. But there is a dipole term, so noise

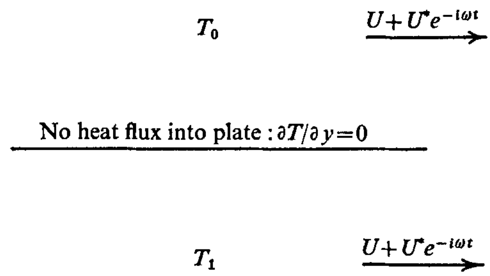

FIGURE 2. Mixing of two streams with different temperatures.

radiated is smaller by the compactness ratio $(\omega\kappa/c^2)^{\frac{1}{2}}$ than the monopole noise induced by heat transfer between the fluid and its boundaries.

As a model problem to illustrate this phenomenon, we consider uniform inviscid flow U in the ξ_1 direction over the semi-infinite plate $\xi_2 = 0$, $\xi_1 < 0$. The temperature of the fluid far upstream is specified as T_0 above the plate and T_1 below the plate, and there is no heat flux into or out of the plate. To introduce unsteadiness into the problem, as before, we allow the plate to judder slightly in the ξ_1 direction with velocity $-U^*e^{-i\omega t}$, so that it occupies the region $\xi_2 = 0$, $\xi_1 < -iU^*e^{-i\omega t}/\omega$. Since, again, at low enough frequencies the source region will be compact, we neglect pressure variations in determining the fluid temperature in the inner region. In the frame of reference in which the plate is at rest (figure 2), the steady part of the temperature satisfies (12),

$$U\partial T'/\partial x = \kappa\nabla^2 T',$$

and the fluctuating part satisfies (13),

$$\partial T^*/\partial t + U\,\partial T^*/\partial x - \kappa\nabla^2 T^* = -U^*e^{-i\omega t}\,\partial T'/\partial x,$$

subject to the boundary conditions

$$\partial T'/\partial y = \partial T^*/\partial y = 0 \quad \text{on} \quad y = 0, \quad x < 0$$

and the upstream conditions

$$T' \to \begin{cases} T_0 & \text{as} \quad x \to -\infty, \quad y > 0, \\ T_1 & \text{as} \quad x \to -\infty, \quad y < 0, \end{cases}$$

and
$$T^* \to 0 \quad \text{as} \quad x \to -\infty.$$

We can deduce the sound radiated directly as before. From Lighthill's acoustic analogy

$$(\rho-\rho_0)(\boldsymbol{\xi},t) = \frac{\xi_i\,\xi_j}{4\pi c^4|\boldsymbol{\xi}|^3}\iiiint \frac{\partial}{\partial\tau}\left\{\frac{1}{(1-M_r)}\frac{\partial}{\partial\tau}\left[\frac{T_{ij}(\mathbf{x}_0,\tau)}{(1-M_r)}\right]\right\}\delta(\tau - t + r/c)\,dV_{\mathbf{x}_0}\,d\tau,$$

where
$$M_r = -(U/c + U^*e^{-i\omega\tau}/c)\,\xi_1/|\boldsymbol{\xi}|, \quad r = |\boldsymbol{\xi} - \boldsymbol{\xi}_0(\mathbf{x}_0,\tau)|$$

and $T_{ij} = c^2\rho_0(T-T_0)\,\delta_{ij}/T_0$, T being the 'inner solution' for the temperature. We thus deduce (see appendix C) the far-field acoustic pressure to be

$$p = \frac{\rho_0 c^2}{(1+M\cos\theta)^2}\frac{T_1-T_0}{T_0}\frac{\tilde{\omega}\kappa}{c^2}\frac{U^*}{c}\frac{\sin\theta\exp\{-i\tilde{\omega}(t-R/c)-\frac{1}{4}i\pi\}}{(8\pi\tilde{\omega}R/c)^{\frac{1}{2}}}, \tag{17}$$

with a relative error $O(M^2)+O(U^*/c)+O(\omega\kappa/c^2)+O((T_1-T_0)/T_0)$, where $M = U/c$ and $\tilde{\omega}$ is the Doppler-shifted frequency $\omega/(1+M\cos\theta)$. The pressure is smaller by the compactness ratio $(\tilde{\omega}\kappa/c^2)^{\frac{1}{2}}$ than that estimated in (16), for flow over a heated semi-infinite plate. The intensity scales with mean velocity to an exponent one greater than when solid constant-temperature surfaces are present, i.e. as U^4 in three dimensions, U^3 in two dimensions and U^2 in one dimension. But again the field is weak because the Péclet number appears in the denominator of the expression for the radiated sound.

We have similarly evaluated (in appendix D) the field in the three-dimensional problem with axial symmetry; the temperature satisfies the boundary condition

$$\partial T/\partial r = 0 \quad \text{on} \quad r = a, \quad \xi < -iU^*(e^{-i\omega t}-1)/\omega$$

and the upstream conditions

$$T \to \begin{cases} T_0 & \text{as} \quad \xi \to -\infty, \quad r > a, \\ T_1 & \text{as} \quad \xi \to -\infty, \quad r < a. \end{cases}$$

The resulting far-field pressure satisfies

$$p \approx \frac{-\rho_0 c^2}{(1+M\cos\theta)^2}\frac{T_1-T_0}{T_0}\frac{\tilde{\omega}\kappa}{c^2}\frac{U^*}{c}\frac{\tilde{\omega}a\sin^2\theta}{c}\frac{\exp\{-i\omega(t-R/c)\}}{4R/a}, \tag{18}$$

providing the frequency ω is low enough. An additional term $\tilde{\omega}a\sin\theta/c$ is introduced because the source is now a quadrupole, the further phase cancellation being due to the axial symmetry.

In deducing these results we have deliberately ignored the scattering of the acoustic waves by the plate or duct. Scattering would augment our estimate of the field, which consequently constitutes only a lower bound on the noise. The detailed calculation of the scattered field would be a long exercise though there is no difficulty of principle in its determination using the method of matched expansions.

If the flow is turbulent rather than laminar, and so is neither two-dimensional nor axisymmetric, we expect the radiated sound to differ in a number of ways. First, we expect real turbulence to be continually regenerating sharp interfaces between the hot and cold fluid, so that, while in the model problem the source region is localized to within a distance downstream from the edge of about $(\kappa/\omega)^{\frac{1}{2}}$, this region will now extend to a distance of about U/ω, the characteristic scale of the flow. The source strength is therefore increased by an amount of the order of the usually large factor $(\omega\kappa/U^2)^{-\frac{1}{2}}$. Second, in the three-dimensional case, we expect the phase cancellation due to the symmetry of the problem to disappear. Finally, any pronounced directivity of the source will probably be weakened. Thus we expect the far-field pressures (17) and (18) to be modified to scale as

$$p \sim \rho_0 c^2 \frac{T_J-T_0}{T_0}\left(\frac{\omega\kappa}{c^2}\right)^{\frac{1}{2}}\left(\frac{U}{c}\right)^2\frac{\exp\{-i\omega(t-R/c)\}}{(\omega R/c)^{\frac{1}{2}}} \tag{19}$$

for the two-dimensional plate problem of a hot stream with temperature T_J and velocity U mixing turbulently with a cold stream of temperature T_0, and as

$$p \sim \rho_0 c^2 \frac{T_J-T_0}{T_0}\left(\frac{\omega\kappa}{c^2}\right)^{\frac{1}{2}}\left(\frac{U}{c}\right)^2\frac{\exp\{-i\omega(t-R/c)\}}{R/D} \tag{20}$$

for the equivalent axisymmetric duct problem when the hot jet has diameter D.

Although we have shown that heat diffusion can generate a far field that depends only weakly on the mean velocity U, we cannot thus account rationally for the correlation obtained by Lush & Fisher (1973). The emergence of an additional length scale (the diffusion length scale $(\kappa/\omega)^{\frac{1}{2}}$) admits an additional compactness ratio $(\omega\kappa/c^2)^{\frac{1}{2}}$; in most flows of interest this is much smaller than the normal compactness ratio M. Thus, although we have a change from the normal U^6 scaling for dipole noise in three dimensions, this is achieved by the introduction of inverse powers of the Péclet number; hence the intensity scales as $M^6 Pe^{-1}$ when the flow is turbulent. Even at the lowest jet velocities considered by Lush & Fisher, the Péclet number is high enough to render these heat-diffusion sources utterly negligible.

6. Nonlinear diffusion as an acoustic source: constant specific heats

Some authors have manipulated Lighthill's source term, or the source term of alternative analogies, apparently to demonstrate the existence of acoustic monopoles (Lilley 1973; Mani 1976*a, b*). We prefer to use a direct approach and start from an expression for the monopole source strength

$$\rho_0 \int_S \mathbf{u}.\mathbf{dS},$$

S being a control surface bounding the compact source region. We examine this expression in some detail because our conclusions differ from those of Crighton (1975), who predicts a substantial monopole source with strength proportional to the jet velocity.† Then the intensity of the resulting sound field would scale as U^4 in three dimensions and be independent of the Reynolds and Péclet numbers.

We return to the nonlinear equations of viscous motion for a perfect gas, (2)–(5). To aid comparison with other authors' work, we introduce the specific entropy s, which satisfies the thermodynamic relation $T ds = c_v dT + p d(1/\rho)$. The temperature equation (5) then becomes the entropy equation

$$\rho T \frac{Ds}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right). \tag{21}$$

The monopole source strength is given by

$$\rho_0 \int_S \mathbf{u}.\mathbf{dS} = \rho_0 \int_V \nabla.\mathbf{u}\, dV = \rho_0 \int_V -\frac{1}{\rho}\frac{D\rho}{Dt} dV$$
$$= \rho_0 \int_V \left(\frac{1}{c_p}\frac{Ds}{Dt} - \frac{1}{\gamma p}\frac{Dp}{Dt}\right) dV,$$

† Our definition of source strength differs from that of some other authors, e.g. Obermeier (1975). We follow Morse & Ingard (1968) and define the monopole source strength density to be Q according to the equation $c^{-2}\partial^2 p/\partial t^2 - \nabla^2 p = \partial Q/\partial t$. In three dimensions, at a distance R the pressure p scales as UD^2Q/R for frequencies $\omega \sim U/D$, D being the length scale of the source region. Because the rate of change of the monopole source strength is the forcing term, the far-field intensity scales as U^4 for a source strength scaling as U. Obermeier defines the source strength density to be $\partial Q/\partial t$, and so his $O(M^2)$ monopole is equivalent to our $O(M)$ monopole in the dependence of its field on jet velocity.

where V is the source region and S is the surface bounding it. Since pressure variations scale as $\rho_0 U^2$ the term

$$\rho_0 \int_V -\frac{1}{\gamma p}\frac{Dp}{Dt} dV$$

will yield a monopole whose strength scales as U^3. We therefore, like Crighton, restrict attention to

$$\rho_0 \int_V \frac{1}{c_p}\frac{Ds}{Dt} dV.$$

Unlike Crighton, however, we proceed to show that this term cannot yield a monopole whose strength is proportional to U unless the specific heats vary.

Using (21) we have

$$\rho_0 \int_V \frac{1}{c_p}\frac{Ds}{Dt} dV = \rho_0 \int_V \left\{\frac{\tau_{ij}}{\rho T c_p}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\frac{Rk}{pc_p}\frac{\partial T}{\partial x_i}\right) + \frac{kR}{p^2 c_p}\frac{\partial p}{\partial x_i}\frac{\partial T}{\partial x_i}\right\} dV \qquad (22)$$

if the specific heats are constant. The second term on the right-hand side of this equation may be transformed by the divergence theorem into a surface integral that vanishes. The first and third terms we examine by a dimensional analysis. For a hot jet with temperature T_J, velocity U and diameter D issuing into a cold still atmosphere with temperature T_0, near-field pressure variations scale as $\rho_0 U^2$ and frequencies as U/D. But there are two length scales involved: the scale for the volume integration is given by the jet diameter D, whilst that for the spatial gradients (for turbulent flows) is given by the diffusion length scale $(\kappa/\omega)^{\frac{1}{2}}$ or $(\kappa D/U)^{\frac{1}{2}}$, or some similar expression involving the kinematic viscosity ν. For a Prandtl number of order unity, the first and third terms on the right-hand side of (22) therefore scale as

$$\rho_0 U^3 D^2/(T_0 c_p), \quad \rho_0 U^3 D^2 (T_J - T_0)/(RT_0^2)$$

(since temperature differences scale as $T_J - T_0$); both these terms yield only weak monopoles whose strengths scale as U^3. So we conclude that no significant monopole source exists.

Crighton correctly deduces that

$$\int_V \rho \frac{Ds}{Dt} dV = \int_V \left\{\frac{\rho\Phi}{T} + \frac{k(\nabla T)^2}{T^2}\right\} dV,$$

where Φ is the dissipation function, and claims that the terms on the right-hand side of this equation are independent of the Reynolds number, since the spatial gradients become large in high Reynolds number turbulent flows. We can deduce this result from the dimensional analysis above, for

$$\int_V \frac{k(\nabla T)^2}{T^2} dV$$

will scale as $\rho_0 U_J c_p (T_J - T_0)^2 D^2/T_0^2$, which is independent of the Reynolds number. But Crighton then argues that

$$\rho_0 \int_V \frac{Ds}{Dt} dV$$

will scale as $\rho_0 U(S_J - S_0) D^2$. This is not so, for, as we have already shown,

$$\rho_0 \int_V \frac{Ds}{Dt} dV$$

does not yield a monopole whose strength scales as U if the specific heats are constant.

Thus, as in the linear case, we find that the only significant diffusion source term is of dipole type unless the specific heats vary with temperature. Obermeier (1975) and Morfey (1976) have also reached this conclusion.

7. Nonlinear diffusion as an acoustic source: variable specific heats

Whenever the specific heats are constant, no monopole results from heat diffusion in a jet that is capable of generating a far field with intensity scaling as U^4. But variations in the specific heats can lead to such a monopole. This can be modelled by allowing γ to vary in our previous analysis, thus introducing an additional term in (22). The monopole source strength therefore becomes

$$\rho_0 \int_S \mathbf{u} . d\mathbf{S} = \rho_0 \int_V \frac{k}{p} \frac{\partial T}{\partial x_i} \frac{\partial}{\partial x_i} \left(\frac{1}{\gamma}\right) dV + O(M^3) \tag{23}$$

(see Morfey 1976). To examine the consequences of this term we first discuss the principal mechanism that causes specific-heat variations in non-monatomic perfect gases.

Energy can be stored in several different modes of molecular motion. If the modes are all fully excited the total energy is distributed equally between them, but quite frequently some modes are unable to hold their full quota because of quantum restrictions. For air at the temperatures of interest to us only the kinetic energy of molecular translation and rotation and the energy associated with molecular vibration are significant.

When a perfect gas is in equilibrium the energy contained in each of the possible modes is a function of the temperature alone. Indeed the translational kinetic energy defines the temperature, so that the specific energy in the three translational modes satisfies

$$E_T(T) = \tfrac{3}{2} RT. \tag{24}$$

At temperatures above a few degrees Kelvin the rotational modes are fully excited, and for diatomic molecules the specific energy in these two modes satisfies

$$E_R(T) = RT. \tag{25}$$

The vibrational mode, however, is fully excited only at temperatures of several thousand degrees Kelvin. At lower temperatures it is unable to hold its full share of the total molecular energy. Except at very high temperatures, it is reasonable to assume that the molecular vibration is harmonic and that it is separable from the molecular rotation. The specific energy in the vibrational mode for diatomic molecules then satisfies

$$E_V(T) = R[\tfrac{1}{2} T_V + T_V/(e^{T_V/T} - 1)] \tag{26}$$

(see Landau & Lifshitz 1958, p. 136), where T_V is called the characteristic temperature for vibration. For temperatures greatly in excess of T_V the vibrational mode is fully excited with $E_V(T) = RT$. The total specific energy is given by $E(T) = E_T(T) + E_R(T) + E_V(T)$ if we neglect the energy in the other modes of molecular motion.

For diatomic gases in equilibrium, the specific heat at constant volume, given by

$$c_v(T) = \frac{dE(T)}{dT} = \tfrac{5}{2}R + \left(\frac{T_V}{T}\right)^2 \frac{Re^{T_V/T}}{(e^{T_V/T}-1)^2},$$

varies because the vibrational energy does not depend linearly on the temperature. In air at temperatures below about 500 °K, the variations are dominated by the contribution from the oxygen molecules, since for oxygen the characteristic temperature for vibration is about 2228 °K while for nitrogen it is about 3336 °K (Herzfeld & Litovitz 1959, table 50). At higher temperatures, however, the contribution from the nitrogen molecules dominates. For temperatures much less than the characteristic temperature for vibration, variations in the specific heats are small, but nevertheless they can lead to a significant monopole source.

To illustrate this source we consider adiabatic mixing at constant pressure of two equal masses of a perfect gas. The temperatures, volumes and densities of the two masses of gas before mixing are T_1, V_1, ρ_1 and T_2, V_2, ρ_2 with $\rho_1 V_1 = \rho_2 V_2$ and $R\rho_1 T_1 = R\rho_2 T_2$. The fully mixed state is specified by T, V, ρ. The governing equations are

$$R\rho_1 T_1 = R\rho_2 T_2 = R\rho T \quad \text{(constant pressure)},$$

$$\rho_1 V_1 + \rho_2 V_2 = \rho V \quad \text{(conservation of mass)}$$

and

$$\rho_1 V_1 E(T_1) + \rho_2 V_2 E(T_2) = \rho V E(T) + R\rho T(V - V_1 - V_2) \quad \text{(conservation of energy)},$$

from which it follows that

$$(V - V_1 - V_2)/(V_1 + V_2) = [2T - (T_1 + T_2)]/(T_1 + T_2),$$

where T satisfies

$$2(RT + E(T)) = RT_1 + E(T_1) + RT_2 + E(T_2).$$

In general $2T \neq T_1 + T_2$, and there will be a change in the total volume as the gases mix. Acoustically, this is equivalent to a monopole.

When a gas is not in equilibrium, the energy contained in any particular mode of molecular motion is not in general a function just of the temperature. The translational kinetic energy always defines the temperature, so that even in non-equilibrium situations the translational specific energy e_T equals its equilibrium value $E_T(T)$:

$$e_T = E_T(T) = \tfrac{3}{2}RT. \tag{27}$$

But variations in the energy contained in other modes will tend to lag behind variations in temperature. The lag between the specific rotational energy e_R and its equilibrium value $E_R(T)$ is small and can be modelled by introducing a bulk viscosity into the viscous stress tensor τ_{ij} (see Lighthill 1956). Then

$$e_R = E_R(T) = RT. \tag{28}$$

The time lag before the specific vibrational energy e_V reaches its equilibrium value, however, is much greater, and must be modelled for small departures from equilibrium with the aid of a rate equation

$$De_V/Dt = (E_V(T) - e_V)/\tau_V \tag{29}$$

(see Lighthill 1956), where τ_V is the relaxation time for the vibrational mode. For oxygen $\tau_V \approx 3\times10^{-3}$ s when $T = 288$ °K and $\tau_V \approx 10^{-4}$ s when $T = 900$ °K; for nitrogen τ_V is not significantly different (Herzfeld & Litovitz 1959, table 50). Introducing water vapour greatly reduces the relaxation time; for example, fully saturated air at 293 °K has a relaxation time of about $0{\cdot}5\times10^{-6}$ s (Lighthill 1956).

We find it convenient to use in our analysis the specific energy

$$e = e_T + e_R + e_V = \tfrac{5}{2}p/\rho + e_V. \tag{30}$$

Instead of the temperature equation (5) or the entropy equation (21) we have the energy equation

$$\rho\frac{De}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = -\frac{\partial q_k}{\partial x_k} + \tau_{ij}\frac{\partial u_i}{\partial x_j} \tag{31}$$

(see Lighthill 1956). The energy flux vector $\mathbf{q}$ has a contribution from the diffusion of molecules with different vibrational energies, so that

$$q_k = -k\,\partial T/\partial x_k - n\mathscr{D}\,\partial e_V/\partial x_k,$$

where $\mathscr{D}$ is the coefficient of self-diffusion and n is the molecular number density (see Clarke & McChesney 1964, §5.9).

The monopole source strength is therefore

$$\rho_0\int_S \mathbf{u}.d\mathbf{S} = \rho_0\int_V\left(-\frac{5}{7}\frac{1}{p}\frac{Dp}{Dt} - \frac{2}{7}\frac{1}{RT}\frac{De_V}{Dt} - \frac{2}{7}\frac{1}{p}\frac{\partial q_k}{\partial x_k} + \frac{2}{7}\frac{\tau_{ij}}{p}\frac{\partial u_i}{\partial x_j}\right)dV. \tag{32}$$

The only significant difference between this and our earlier result (22) is the additional term

$$\rho_0\int_V -\frac{2}{7}\frac{1}{RT}\frac{De_V}{Dt}\,dV.$$

We may again neglect all the other terms on the right-hand side of (32), so that, correct to $O(M^2)$,

$$\rho_0\int_S \mathbf{u}.d\mathbf{S} = \rho_0\int_V -\frac{2}{7}\frac{1}{RT}\frac{De_V}{Dt}\,dV. \tag{33}$$

The vibrational mode effectively acts as a source and sink of heat, causing the fluid to expand and contract and generating a monopole sound field. The source term does not in general vanish since, when two masses of a non-monatomic gas mix, the vibrational energy lost by the hotter gas exceeds that gained by the colder.

At high frequencies, when the time scale of the fluctuations is much smaller than the relaxation time, the vibrational mode is unable to respond to the rapid fluctuations in the temperature, and we have frozen flow with the rate equation (29) reducing to $De_V/Dt = 0$. Then the monopole source strength (33) vanishes.

At low frequencies, the rate equation reduces to $E_V(T) - e_V = 0$; i.e. the energy in the vibrational mode always equals its equilibrium value. In this case the monopole source strength (33) becomes

$$\rho_0 \int_S \mathbf{u}.\mathbf{dS} = \rho_0 \int -\frac{2}{7}\frac{1}{RT}\frac{DE_V}{Dt}dV.$$

Since E_V is a function of temperature alone, $DE_V/Dt = dE_V/dT\, DT/Dt$. From the energy equation (21) and equations (27) and (28), therefore,

$$\frac{DE_V}{Dt} = \frac{dE_V}{dT}\frac{2}{7}\frac{1}{Rp}\left\{\frac{Dp}{Dt} - \rho\frac{De_V}{Dt} - \frac{\partial q_k}{\partial x_k} + \tau_{ij}\frac{\partial u_i}{\partial x_j}\right\}.$$

At temperatures much less than the characteristic temperature for vibration, nearly all the molecular energy is contained in the translational and rotational modes, so we can neglect terms quadratic in the specific heat dE_V/dT for the vibrational mode. Thus

$$\rho_0 \int_S \mathbf{u}.\mathbf{dS} = \rho_0 \int_V \frac{-1}{c_p^2 \rho T}\frac{dE_V}{dT}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)dV + O(M^3) + O\left(\frac{1}{R^2}\left(\frac{dE_V}{dT}\right)^2\right).$$

Further manipulation gives the low frequency monopole source strength as

$$\rho_0 \int_S \mathbf{u}.\mathbf{dS} = \rho_0 \int_V \frac{\kappa}{c_p T}\frac{d^2E_V}{dT^2}\left(\frac{\partial T}{\partial x_i}\right)^2 dV + O(M^3) + O\left(\frac{1}{R^2}\left(\frac{dE_V}{dT}\right)^2\right). \quad (34)$$

These high and low frequency results can be obtained directly from (23). If the temperature fluctuations are small and periodic in time with radian frequency ω, then the effective specific heats are given by

$$(c_v)_{\text{eff}} = \tfrac{5}{2}R + (dE_V/dT)/(1 + i\omega\tau_V),$$
$$(c_p)_{\text{eff}} = \tfrac{7}{2}R + (dE_V/dT)/(1 + i\omega\tau_V)$$

(see Herzfeld & Litovitz 1959; Lighthill 1956). At low frequencies, $\omega\tau_V \ll 1$,

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\gamma_{\text{eff}}}\right) = \frac{\partial}{\partial x_i}\left(\frac{(c_v)_{\text{eff}}}{(c_p)_{\text{eff}}}\right) = \frac{R}{c_p^2}\frac{\partial T}{\partial x_i}\frac{d^2E_V}{dT^2},$$

neglecting terms smaller by factors of $R^{-1}dE_V/dT$, and the monopole source strength is given by (34). At high frequencies, $\omega\tau_V \gg 1$, the vibrational energy cannot respond to the variations in temperature, so $\partial\gamma_{\text{eff}}^{-1}/\partial x_i = 0$ and the monopole source strength vanishes.

Dimensional arguments like those in § 6 suggest that the low frequency monopole source strength should scale as

$$[\rho_0 (T_J - T_0)^2\, U\delta\, d^2E_V(T_0)/dT^2]/(c_p T_0)$$

for the problem of a hot two-dimensional stream with temperature T_J and velocity U mixing turbulently with a cold stream of temperature T_0, where δ is the thickness of the turbulent shear layer, and that the corresponding expression for the equivalent axisymmetric problem is

$$[\rho_0 (T_J - T_0)^2 U D^2 d^2E_V(T_0)/dT^2]/(c_p T_0)$$

when the hot jet has diameter D. The radiated far-field pressures at a distance R scale respectively as

$$p \sim \rho_0 c^2 \frac{T_0}{c_p} \frac{d^2E_V(T_0)}{dT^2} \frac{(T_J - T_0)^2}{T_0^2} \left(\frac{U}{c}\right)^2 \left(\frac{c}{\omega R}\right)^{\frac{1}{2}} \tag{35}$$

and

$$p \sim \rho_0 c^2 \frac{T_0}{c_p} \frac{d^2E_V(T_0)}{dT^2} \frac{(T_J - T_0)^2}{T_0^2} \left(\frac{U}{c}\right)^2 \frac{D}{R}. \tag{36}$$

For the two-dimensional problem of §5 for laminar inviscid flow, we deduce in appendix E that at low frequencies the noise radiated from the nonlinear monopole source associated with variations in the specific heats has fluctuating pressure

$$p = \rho c^2 \left(\frac{\omega\kappa}{c^2}\right)^{\frac{1}{2}} \frac{(T_1 - T_0)^2}{T_0^2} \frac{U^*}{c} \frac{T_0}{c_p} \frac{d^2E_V(T_0)}{dT^2} \frac{\exp\{-i\omega(t - R/c) - \frac{1}{4}i\pi\}}{(16\pi\omega R/c)^{\frac{1}{2}}}. \tag{37}$$

We have neglected terms smaller by factors of M, the Mach number, of $\omega\kappa/U^2$, the reciprocal of the Péclet number, of $R^{-1}dE_V/dT$, the relative specific heat for the vibrational mode, or of $(T_1 - T_0)/T_0$, the relative temperature difference. For the problem of turbulent flow, as in §5, we expect the pressure to be increased by an amount of the order of the normally large factor $(\omega\kappa/U^2)^{-\frac{1}{2}}$, and we recover the result (35) obtained by dimensional analysis. We have not examined the problem of laminar axisymmetric flow in this manner because the mathematics become very involved; presumably similar conclusions would hold.

We conclude that variability of the specific heats admits a monopole source in hot turbulent jets that can generate a far field scaling as U^4 in three dimensions. The strength of this monopole source has a distinct frequency dependence, vanishing at high frequencies. It is much weaker, even at low frequencies, than the conclusions of Crighton would suggest. Obermeier's published estimate of the source strength is also in error; it should be smaller by the factor

$$(T/c_p)\, d^2E_V/dT^2 \approx 2 \times 10^{-2},$$

or 34 dB typically. But even so this monopole will normally dominate over the dipole mechanism of §5 [compare (35) and (36) with (19) and (20) or (37) with (17)], since typically $(\omega\kappa/c^2)^{\frac{1}{2}} \approx 10^{-3}$.

8. Conclusions

We have determined the sound radiated from an unsteady heat source. A monopole contribution dominates provided that the source is compact, and the radiated sound may then be deduced from a knowledge of the heat input alone.

Estimates of the sound radiated due to fluctuating heat input are compared with estimates of that due to fluctuating forces for certain aeroengine and underwater applications. Although the intensity of sound due to the viscous boundary-layer drag is smaller by the Mach number squared than that due to the heat input, the fluctuating lift or form drag is often large enough to produce the most noise. But when boundary layers are turbulent heat sources are probably significant.

In the absence of heat transfer between the fluid and any foreign bodies it is shown that the diffusive noise source is of dipole type provided only that the specific heats remain constant. A more significant source is associated with variability of the specific heats; i.e. a monopole with far-field intensity scaling as

Mach number to the fourth power. The monopole source strength is independent of Reynolds number, but it is multiplied by another small parameter (the rate of change of the specific heats with temperature). The monopole is associated with the flow of heat into and out of the vibrational energy mode of molecular motion; relaxation effects restrict the response of the vibrational degree of freedom to rapid fluctuations in temperature, so we argue that any noise radiated from this source will be of low frequency. Model jet experiments do appear to show this kind of frequency dependence (see Hoch *et al.* 1973).

Lush & Fisher have obtained a correlation of hot-jet noise data by assuming that the increase with temperature of the low frequency noise at low Mach numbers is due to an additional diffusive monopole source. If this assumption is correct there will be an unusual scale effect. The presence of a second time scale (the relaxation time for the vibrational energy mode) means that acoustic frequencies will not simply scale as U/D. In addition there will be a marked variability in the spectral distribution when the air is humid. The relaxation time for the vibrational mode depends critically on the humidity (about 10^{-3} s for dry air, but 10^{-6} s or less for humid air), and consequently so does the relaxation frequency (about 150 Hz for dry air, but 150 kHz or greater for humid air). The monopole source is significant at frequencies much less than the relaxation frequency and so could be significant even at high frequencies if the jet were humid. Although these idiosyncrasies could be helpful in estimating the importance of heat diffusion as a noise source, they might present great difficulties in the application of results for model jets to full-scale engine configurations, and may in fact be the cause of some of the scatter in the available experimental data.

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Appendix A

The steady temperature is given by the solution of (12),

$$U\partial T'/\partial x = \kappa\nabla^2 T',$$

with $$T' = (T_1 - T_0)\{H(x) - H(x-a)\} \quad \text{on} \quad y = 0.$$

The substitution $\phi = T'e^{ikx}$, where $k = iU/2\kappa$, gives

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + k^2\phi = 0, \tag{38}$$

with $$\phi(x,0) = (T_1 - T_0)\{H(x) - H(x-a)\}e^{ikx}.$$

Taking Fourier transforms in x defined by

$$\Phi(\alpha, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,y)\, e^{i\alpha x}\, d\alpha,$$

we have the solution

$$\Phi(\alpha, y) = \frac{-(T_1 - T_0)}{2\pi i(\alpha + k)}\{1 - e^{ia(\alpha+k)}\}\, e^{-\zeta y},$$

where $\zeta = (\alpha^2 - k^2)^{\frac{1}{2}}$ is taken to have positive real part. $\Phi(\alpha, y)$ is analytic in the strip $U/2k > \operatorname{Im}(\alpha) > -U/2k$, and

$$\phi(x, y) = \int_{-\infty}^{\infty} \Phi(\alpha, y)\, e^{-i\alpha x}\, d\alpha.$$

The unsteady temperature satisfies (13),

$$\partial T^*/\partial t + U\partial T^*/\partial x - \kappa\nabla^2 T^* = -U^* e^{-i\omega t}\, \partial T/\partial x,$$

with $T^* = 0$ on $y = 0$. The substitution $\phi^* = T^* e^{ikx+i\omega t}$ reduces this to

$$\frac{\partial^2\phi^*}{\partial x^2} + \frac{\partial^2\phi^*}{\partial y^2} + l^2\phi^* = e^{ikx}\frac{U^*}{\kappa}\frac{\partial}{\partial x}\{\phi\, e^{-ikx}\} \tag{39}$$

with $\phi^* = 0$ on $y = 0$. Here $l^2 = (iU/2\kappa)^2 + i\omega/\kappa$. Again taking Fourier transforms, we have the solution

$$\phi^*(x, y) = \int_{-\infty}^{\infty} \Phi^*(\alpha, y)\, e^{-i\alpha x}\, d\alpha,$$

where
$$\Phi^*(\alpha, y) = \frac{U^*(T_1 - T_0)}{2\pi i\omega}\{1 - e^{ia(\alpha+k)}\}\{e^{-\zeta y} - e^{-\eta y}\},$$

and $\eta = (\alpha^2 - l^2)^{\frac{1}{2}}$ is taken to have positive real part.

We thus have

$$\begin{aligned} T = T_0 &- \int_{-\infty}^{\infty} \frac{(T_1 - T_0)}{2\pi i(\alpha + k)}(1 - e^{ia(\alpha+k)}) \exp[-i(\alpha + k)x - \zeta y]\, d\alpha \\ &+ \int_{-\infty}^{\infty} \frac{U^*(T_1 - T_0)}{2\pi i\omega}(1 - e^{ia(\alpha+k)})(e^{-\zeta y} - e^{-\eta y}) \exp[-i(\alpha + k)x - i\omega t]\, d\alpha. \end{aligned} \tag{40}$$

So
$$\begin{aligned} \frac{\partial}{\partial\tau}\left\{\frac{T_{ij}(\mathbf{x}, \tau)}{(1 - M_r)}\right\} &= \frac{\partial}{\partial\tau}\left\{\frac{c^2\rho_0(T - T_0)}{T_0(1 + M\cos\theta)}\delta_{ij}\left[1 - \frac{U^*\cos\theta\, e^{-i\omega t}}{c(1 + M\cos\theta)} + O\left(\frac{U^{*2}}{c^2}\right)\right]\right\} \\ &= \int_{-\infty}^{\infty} \frac{-c^2\rho_0\delta_{ij}\, U^*(T_1 - T_0)(1 - e^{ia(\alpha+k)})}{2\pi T_0(1 + M\cos\theta)} \exp[-i(\alpha + k)x - i\omega t] \\ &\qquad \times \left\{e^{-\zeta y} - e^{-\eta y} + \frac{\tilde{\omega}\cos\theta\, e^{-\zeta y}}{c(\alpha + k)}\right\} d\alpha + O\left(\frac{U^{*2}}{c^2}\right), \end{aligned}$$

since $M_r = -(U/c + U^* e^{-i\omega\tau}/c)\cos\theta$, $M = U/c$, $\tilde{\omega} = \omega/(1 + M\cos\theta)$ and θ is the angle of observation. Substituting in the three-dimensional result (14) with the zero suffix on $\mathbf{x}$ omitted,

$$(\rho - \rho_0)(\boldsymbol{\xi}, t) = \frac{\xi_i\xi_j}{4\pi c^4|\boldsymbol{\xi}|^3}\iiiint \frac{\partial}{\partial\tau}\left\{\frac{1}{(1 - M_r)}\frac{\partial}{\partial\tau}\left(\frac{T_{ij}(\mathbf{x}, \tau)}{(1 - M_r)}\right)\right\}\delta(\tau - t + r/c)\, dV_{\mathbf{x}}\, d\tau_0,$$

and integrating with respect to x_{03} by the method of stationary phase, we have, since $\partial p/\partial y = 0$ on $y = 0$,

$$c^2(\rho-\rho_0)(\xi, t) = \frac{i\tilde{\omega}\rho_0 U^*(T_1-T_0)\exp\{-i\tilde{\omega}(t-R/c)+\frac{1}{4}i\pi\}}{2\pi T_0(8\pi\tilde{\omega}R/c)^{\frac{1}{2}}(1+M\cos\theta)^2}$$
$$\times \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}(1-e^{ia(\alpha+k)})\exp\{-i(\alpha+k)x-i\tilde{\omega}x\cos\theta/c\}$$
$$\times\{e^{-i\tilde{\omega}y\sin\theta/c}+e^{i\tilde{\omega}y\sin\theta/c}\}\left\{e^{-\zeta y}-e^{-\gamma y}+\frac{\tilde{\omega}\cos\theta\, e^{-\zeta y}}{c(\alpha+k)}\right\}dx\,dy\,d\alpha+O(U^{*2}/c^2).$$

Thus the far-field acoustic pressure is

$$p = \frac{\rho_0 c^2}{(1+M\cos\theta)^2}\frac{T_1-T_0}{T_0}\left(\frac{\tilde{\omega}\kappa}{c^2}\right)^{\frac{1}{2}}\frac{U^*}{c}(1-e^{-ia\tilde{\omega}\cos\theta/c})\frac{e^{-i\tilde{\omega}(t-R/c)}}{(2\pi\tilde{\omega}R/c)^{\frac{1}{2}}}$$
$$+O(\tilde{\omega}\kappa/c^2)+O(U^{*2}/c^2). \tag{41}$$

For small $\tilde{\omega}a/c$, we obtain (15),

$$p = \frac{i\rho_0 c^2}{(1+M\cos\theta)^2}\frac{T_1-T_0}{T_0}\left(\frac{\tilde{\omega}\kappa}{c^2}\right)^{\frac{1}{2}}\frac{U^*}{c}\frac{\tilde{\omega}a\cos\theta}{c}\frac{\exp[-i\tilde{\omega}(t-R/c)]}{(2\pi\tilde{\omega}R/c)^{\frac{1}{2}}},$$

while, for $a\to\infty$, we obtain (16),

$$p = \frac{\rho_0 c^2}{(1+M\cos\theta)^2}\frac{T_1-T_0}{T_0}\left(\frac{\tilde{\omega}\kappa}{c^2}\right)^{\frac{1}{2}}\frac{U^*}{c}\frac{\exp[-i\tilde{\omega}(t-R/c)]}{(2\pi\tilde{\omega}R/c)^{\frac{1}{2}}}.$$

Appendix B

We now solve (38),

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+k^2\phi = 0,$$

subject to $\phi = (T_1-T_0)e^{ikx}$ for $y = 0$, $x > 0$, ϕ continuous and $\partial\phi/\partial y = 0$ for $y = 0$, $x < 0$, and subject to radiation and edge conditions. Taking Fourier transforms, as before, we have $\Phi(\alpha, y) = A(\alpha)e^{-\zeta|y|}$ (since ϕ is continuous on $y = 0$).

Using the notation of Noble (1958), i.e. denoting

$$\frac{1}{2\pi}\int_0^{\infty}\phi(x,y)e^{i\alpha x}dx \quad \text{by} \quad \Phi_+(\alpha, y)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{0}\phi(x,y)e^{i\alpha x}dx \quad \text{by} \quad \Phi_-(\alpha, y),$$

we have

$$\Phi_+(\alpha, 0^+) = \Phi_+(\alpha, 0^-) = -(T_1-T_0)/\{2\pi i(\alpha+k)\},$$
$$\Phi_-(\alpha, 0^+) = \Phi_-(\alpha, 0^-)$$

and

$$\Phi'_-(\alpha, 0^+) = \Phi'_-(\alpha, 0^-) = 0.$$

Thus $\Phi'_+(\alpha, 0^+)-\Phi'_+(\alpha, 0^-) = -2\zeta A(\alpha) = -2\zeta\{\Phi_-(\alpha, 0)-(T_1-T_0)/[2\pi i(\alpha+k]\}$.

Since $\Phi_+(\alpha, y)$ is regular for $\text{Im}\,\alpha > -U/2\kappa$, $\Phi_-(\alpha, y)$ is regular for $\text{Im}\,\alpha < U/2\kappa$, and ζ is regular and non-zero in the strip $U/2\kappa > \text{Im}\,\alpha > -U/2\kappa$, we may solve the equation by the Wiener–Hopf technique. We have

$$\{\Phi'_+(\alpha,0^+)-\Phi'_+(\alpha,0^-)\}\frac{1}{(\alpha+k)^{\frac{1}{2}}}-\frac{(T_1-T_0)(-2k)^{\frac{1}{2}}}{\pi i(\alpha+k)}$$
$$=-2(\alpha-k)^{\frac{1}{2}}\Phi_-(\alpha,0)+\frac{(T_1-T_0)}{\pi i(\alpha+k)}\{(\alpha-k)^{\frac{1}{2}}-(-2k)^{\frac{1}{2}}\}=J(\alpha),$$

where $J(\alpha)$ is a function regular in the whole complex plane. Close to the edge $\nabla^2T\approx 0$ and so we have the edge conditions $\phi=O(1)$ and $\nabla\phi=O(|\mathbf{x}|^{-m})$ with $m<1$ (see Noble 1958, p. 74). Consequently

$$\Phi_-(\alpha,0)=O(\alpha^{-1})\quad\text{and}\quad\Phi'_+(\alpha,0^+)-\Phi'_+(\alpha,0^-)=O(\alpha^{m-1}),$$

so $J(\alpha)\to 0$ as $|\alpha|\to\infty$ in both the upper and the lower half-plane; by Liouville's theorem $J(\alpha)\equiv 0$. It follows that

$$A(\alpha)=-(T_1-T_0)(-2k)^{\frac{1}{2}}/\{2\pi i(\alpha+k)(\alpha-k)^{\frac{1}{2}}\},$$

so

$$\phi(x,y)=\int_{-\infty}^{\infty}\frac{-(T_1-T_0)(-2k)^{\frac{1}{2}}}{2\pi i(\alpha+k)(\alpha-k)^{\frac{1}{2}}}\exp(-\zeta|y|-i\alpha x)\,d\alpha.$$

Equation (39), $$\frac{\partial^2\phi^*}{\partial x^2}+\frac{\partial^2\phi^*}{\partial y^2}+l^2\phi^*=e^{ikx}\frac{U^*}{\kappa}\frac{\partial}{\partial x}\{\phi\, e^{-ikx}\},$$

with ϕ^* zero on $y=0$, $x>0$, ϕ^* continuous and $\partial\phi^*/\partial y$ zero on $y=0$, $x<0$ and subject to radiation and edge conditions, may be solved by inspection:

$$\phi^*(x,y)=\int_{-\infty}^{\infty}\frac{U^*(T_1-T_0)(-2k)^{\frac{1}{2}}}{2\pi i\omega}e^{-i\alpha x}\left\{\frac{e^{-\zeta|y|}}{(\alpha-k)^{\frac{1}{2}}}-\frac{e^{-\eta|y|}}{(\alpha-l)^{\frac{1}{2}}}\right\}d\alpha.$$

We verify only that this solution satisfies the boundary conditions. For $y=0$, $x>0$, we may complete the contour in the lower half-plane to obtain $\phi^*=0$. For $y=0$, $x<0$, ϕ^* is continuous, and we may complete the contour in the upper half-plane to obtain $\partial\phi^*/\partial y=0$ (as $|\alpha|\to\infty$ the integrand $\sim\alpha^{-\frac{1}{2}}e^{-i\alpha x}$, so that the contribution from the semicircle at infinity vanishes by Jordan's lemma). It is not difficult to verify that the governing equation is obeyed, and that the radiation and edge conditions are satisfied (see, for example, Noble 1958, pp. 72–73).

Thus we have

$$\begin{aligned}T=T_0&-\int_{-\infty}^{\infty}\frac{(T_1-T_0)(-2k)^{\frac{1}{2}}}{2\pi i(\alpha+k)(\alpha-k)^{\frac{1}{2}}}\exp[-\zeta|y|-i(\alpha+k)x]\,d\alpha\\&+\int_{-\infty}^{\infty}\frac{U^*(T_1-T_0)(-2k)^{\frac{1}{2}}}{2\pi i\omega}\exp[-i(\alpha+k)x-i\omega t]\left\{\frac{e^{-\zeta|y|}}{(\alpha-k)^{\frac{1}{2}}}-\frac{e^{-\eta|y|}}{(\alpha-l)^{\frac{1}{2}}}\right\}d\alpha.\end{aligned}\tag{42}$$

So as before

$$\begin{aligned}c^2(\rho-\rho_0)(\boldsymbol{\xi},t)=&\frac{i\tilde{\omega}\rho_0U^*(T_1-T_0)(-2k)^{\frac{1}{2}}\exp\{-i\tilde{\omega}(t-R/c)+\frac{1}{4}i\pi\}}{2\pi T_0(8\pi\tilde{\omega}R/c)^{\frac{1}{2}}(1+M\cos\theta)^2}\\&\times\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\exp\{-i(\alpha+k)x-i\tilde{\omega}x\cos\theta/c\}\\&\qquad\times\{\exp[-i\tilde{\omega}y\sin\theta/c]+\exp[i\tilde{\omega}y\sin\theta/c]\}\\&\times\left\{\frac{e^{-\zeta v}}{(\alpha-k)^{\frac{1}{2}}}-\frac{e^{-\eta v}}{(\alpha-l)^{\frac{1}{2}}}+\frac{\tilde{\omega}\cos\theta\, e^{-\zeta v}}{c(\alpha+k)(\alpha-k)^{\frac{1}{2}}}\right\}dx\,dy\,d\alpha+O\left(\frac{U^{*2}}{c^2}\right),\end{aligned}$$

since $\partial p/\partial y = 0$ on $y = 0$ (by symmetry for $x < 0$). Thus we again obtain (16):

$$p = \frac{\rho_0 c^2}{(1+M\cos\theta)^2}\frac{T_1-T_0}{T_0}\left(\frac{\tilde{\omega}\kappa}{c^2}\right)^{\frac{1}{2}}\frac{U^*}{c}\frac{\exp[-i\tilde{\omega}(t-R/c)]}{(2\pi\tilde{\omega}R/c)^{\frac{1}{2}}} + O\left(\frac{\tilde{\omega}\kappa}{c^2}\right) + O\left(\frac{U^{*2}}{c^2}\right).$$

Appendix C

In the frame of reference (x, y, z) moving with the plate, the temperature on $y = 0, x > 0$ is $\frac{1}{2}(T_0+T_1)$ by symmetry, while on $y = 0$, $x < 0$ we have $\partial T/\partial y = 0$. It follows that (42) may be used to deduce the steady and fluctuating temperature:

$$\begin{aligned} T = {} & \frac{T_0+T_1}{2} - \frac{(T_1-T_0)}{2}\operatorname{sgn} y \\ & - \operatorname{sgn} y \int_{-\infty}^{\infty} \frac{(T_1-T_0)(-2k)^{\frac{1}{2}}\exp[-\zeta|y| - i(\alpha+k)x]}{4\pi i(\alpha+k)(\alpha-k)^{\frac{1}{2}}}\, d\alpha. \\ & + \operatorname{sgn} y \int_{-\infty}^{\infty} \frac{U^*(T_1-T_0)(-2k)^{\frac{1}{2}}}{4\pi i\omega}\exp[-i(\alpha+k)x - i\omega t] \\ & \qquad \times \left\{\frac{e^{-\zeta|y|}}{(\alpha-k)^{\frac{1}{2}}} - \frac{e^{-\eta|y|}}{(\alpha-l)^{\frac{1}{2}}}\right\} d\alpha. \end{aligned} \tag{43}$$

To determine the sound radiated directly, we use (14) and integrate with respect to x_{03} by the method of stationary phase. Thus

$$\begin{aligned} c^2(\rho-\rho_0)(\boldsymbol{\xi}, t) = {} & \frac{i\tilde{\omega}\rho_0 U^*(T_1-T_0)(-2k)^{\frac{1}{2}}\exp\{-i\tilde{\omega}(t-R/c)+\frac{1}{4}i\pi\}}{4\pi T_0(8\pi\tilde{\omega}R/c)^{\frac{1}{2}}(1+M\cos\theta)^2} \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-i(\alpha+k)x - i\tilde{\omega}x\cos\theta/c - i\tilde{\omega}y\sin\theta/c\}\operatorname{sgn} y \\ & \times \left\{\frac{e^{-\zeta|y|}}{(\alpha-k)^{\frac{1}{2}}} - \frac{e^{-\eta|y|}}{(\alpha-l)^{\frac{1}{2}}} + \frac{\tilde{\omega}\cos\theta\, e^{-\zeta|y|}}{c(\alpha+k)(\alpha-k)^{\frac{1}{2}}}\right\} dx\, dy\, d\alpha + O\left(\frac{U^{*2}}{c^2}\right), \end{aligned}$$

which gives (17):

$$\begin{aligned} p = {} & \frac{\rho_0 c^2}{(1+M\cos\theta)^2}\frac{(T_1-T_0)}{T_0}\frac{\tilde{\omega}\kappa}{c^2}\frac{U^*}{c}\sin\theta \\ & \times \frac{\exp[-i\tilde{\omega}(t-R/c) - \frac{1}{4}i\pi]}{(8\pi\tilde{\omega}R/c)^{\frac{1}{2}}} + O\left(\frac{\tilde{\omega}\kappa}{c^2}\right) + O\left(\frac{U^{*2}}{c^2}\right). \end{aligned}$$

Appendix D

The steady temperature in the equivalent three-dimensional axisymmetric problem satisfies $U\partial T'/\partial x = \kappa\nabla^2 T'$ with $\partial T'/\partial r = 0$ on $r = a$, $x < 0$, with $T \to T_0$ as $x \to -\infty$, $r > a$ and with $T \to T_1$ as $x \to -\infty$, $r < a$, and is subject to radiation and edge conditions. Setting

$$T = \begin{cases} \phi e^{-ikx} + T_0 & \text{for} \quad r > a, \\ \phi e^{-ikx} + T_1 & \text{for} \quad r < a, \end{cases}$$

the governing equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial x^2} + k^2\phi = 0, \tag{44}$$

with $\partial\phi/\partial r$ continuous on $r = a, x > 0$ and zero on $r = a, x < 0$, and with

$$\phi(r=a^+) - \phi(r=a^-) = (T_1 - T_0)\, e^{ikx} \quad \text{for} \quad x > 0.$$

Taking Fourier transforms in x, defining

$$\Phi(\alpha, r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x, r)\, e^{i\alpha x}\, dx,$$

we have

$$\Phi(\alpha, r) = \begin{cases} A(\alpha)\, K_0(\zeta r) & \text{for} \quad r > a, \\ B(\alpha)\, I_0(\zeta r) & \text{for} \quad r < a, \end{cases}$$

where $\zeta = (\alpha^2 - k^2)^{\frac{1}{2}}$ has positive real part. Using the notation of Noble, as before, we have

$$\Phi_+(\alpha, a^+) - \Phi_+(\alpha, a^-) = -(T_1 - T_0)/[2\pi i(\alpha + k)],$$

$$\Phi'_+(\alpha, a^+) = \Phi'_+(\alpha, a^-), \quad \Phi'_-(\alpha, a^+) = \Phi'_-(\alpha, a^-) = 0.$$

So

$$A(\alpha)\, K'_0(\zeta a) = B(\alpha)\, I'_0(\zeta a)$$

and

$$\Phi_-(\alpha, a^+) - \Phi_-(\alpha, a^-) - \frac{(T_1 - T_0)}{2\pi i(\alpha + k)} = A(\alpha)\, K_0(\zeta a) - B(\alpha)\, I_0(\zeta a)$$

$$= \Phi'_+(\alpha, a)\left\{\frac{K_0(\zeta a)}{\zeta K'_0(\zeta a)} - \frac{I_0(\zeta a)}{\zeta I'_0(\zeta a)}\right\}.$$

Now

$$\frac{K_0(\zeta a)}{\zeta K'_0(\zeta a)} - \frac{I_0(\zeta a)}{\zeta I'_0(\zeta a)} = \frac{1}{a\zeta^2 I'_0(\zeta a)\, K'_0(\zeta a)} = \frac{-2}{a\zeta^2 K_+(\alpha)\, K_-(\alpha)},$$

where $K_+(\alpha)\, K_-(\alpha) = -2K'_0(\zeta a)\, I'_0(\zeta a)$, and $K_\pm(\alpha)$ are analytic and asymptote to $|\alpha|^{-\frac{1}{2}}$ in the upper and lower half-planes respectively (see Noble 1958, pp. 110–118). Thus

$$\{\Phi_-(\alpha, a^+) - \Phi_-(\alpha, a^-)\}(\alpha - k)\, K_-(\alpha) - \frac{(T_1 - T_0)}{2\pi i(\alpha + k)}\{(\alpha - k)\, K_-(\alpha) + 2kK_-(-k)\}$$

$$= \frac{-2\Phi'_+(\alpha, a)}{(\alpha + k)\, aK_+(\alpha)} - \frac{(T_1 - T_0)}{2\pi i(\alpha + k)}\, 2kK_-(-k) = J(\alpha),$$

where $J(\alpha)$ is analytic in the entire complex plane. Edge conditions ensure that $J(\alpha) \to 0$ as $|\alpha| \to \infty$, so $J(\alpha) \equiv 0$ by Liouville's theorem. Thus

$$A(\alpha) = -(T_1 - T_0)\, UaK_-(-k)\, K_+(\alpha)/\{4\pi\kappa\zeta K'_0(\zeta a)\}$$

and

$$B(\alpha) = -(T_1 - T_0)\, UaK_-(-k)\, K_+(\alpha)/\{4\pi\kappa\zeta I'_0(\zeta a)\}.$$

So

$$\phi(x, r) = \int_{-\infty}^{\infty} \frac{-(T_1 - T_0)\, UaK_-(-k)\, K_+(\alpha)}{4\pi\kappa\zeta}\, \mathscr{C}(\zeta r)\, e^{-i\alpha x}\, d\alpha,$$

where

$$\mathscr{C}(\zeta r) = \begin{cases} K_0(\zeta r)/K'_0(\zeta a) & \text{for} \quad r > a, \\ I_0(\zeta r)/I'_0(\zeta a) & \text{for} \quad r < a. \end{cases}$$

The solution of the unsteady temperature equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi^*}{\partial r}\right) + \frac{\partial^2\phi^*}{\partial x^2} + l^2\phi^* = e^{ikx}\frac{U^*}{\kappa}\frac{\partial}{\partial x}\{\phi\, e^{-ikx}\} \tag{45}$$

which satisfies the boundary conditions $\partial\phi^*/\partial r$ continuous on $r = a, x > 0$ and zero on $r = a, x < 0$ and ϕ^* continuous on $r = a, x > 0$ and which also satisfies the radiation and edge conditions may be found by inspection to be

$$\phi^*(x,r) = \int_{-\infty}^{\infty} \frac{U(T_1-T_0)\,U^*aK_-(-k)}{4\pi\kappa\omega} \times \left\{\frac{(\alpha+\,)\,(K_+(\alpha)\mathscr{C}(\zeta r)}{\zeta} - \frac{(\alpha+l)\,L_\pm(\alpha)\mathscr{C}(\eta r)}{\eta}\right\} e^{-i\alpha x}\,d\alpha,$$

where $\eta = (\alpha^2-l^2)^{\frac{1}{2}}$ is taken to have positive real part.

$$L_+(\alpha)\,L_-(\alpha) = -2K_0'(\eta a)\,I_0'(\eta a)$$

with $L_\pm(\alpha)$ analytic and asymptoting to $|\alpha|^{-\frac{1}{2}}$ in the upper and lower half-planes respectively. Again, as in appendix B, we verify only that the boundary conditions are satisfied. We have

$$\phi^*(r=a^+)-\phi^*(r=a^-) = \frac{U(T_1-T_0)\,U^*aK_-(-k)}{4\pi\kappa\omega} \times \int_{-\infty}^{\infty} \left\{\frac{(\alpha+k)\,K_+(\alpha)}{a\zeta^2 I_0'(\zeta a)\,K_0'(\zeta a)} - \frac{(\alpha+l)\,L_+(\alpha)}{a\eta^2 I_0'(\eta a)\,K_0'(\eta a)}\right\} e^{-i\alpha x}\,d\alpha$$

since
$$\frac{K_0(\zeta a)}{\zeta K_0'(\zeta a)} - \frac{I_0(\zeta a)}{\zeta I_0'(\zeta a)} = \frac{1}{a\zeta^2 I_0'(\zeta a)\,K_0'(\zeta a)}.$$

For $x > 0$ the contour may be completed in the lower half-plane. The integral vanishes so ϕ^* is continuous on $r = a, x > 0$. $\partial\phi^*/\partial r$ is clearly continuous on $r = a$, and is given by

$$\frac{\partial\phi^*}{\partial r}(r=a) = \frac{U(T_1-T_0)\,U^*a\,K_-(-k)}{4\pi\kappa\omega} \times \int_{-\infty}^{\infty} \{(\alpha+k)\,K_+(\alpha)-(\alpha+l)\,L_+(\alpha)\}\,e^{-i\alpha x}\,d\alpha.$$

As $|\alpha|\to\infty$, $K_+(\alpha) \sim (-i\alpha)^{-\frac{1}{2}}$ for Im $(\alpha) > 0$ (see Levine & Schwinger 1948), so the integrand $\sim e^{-i\alpha x}\alpha^{-\frac{1}{2}}$. By Jordan's lemma the contribution from the semicircle at infinity in the upper half-plane vanishes for $x < 0$ and so $\partial\phi^*/\partial r = 0$ for $r = a$, $x < 0$.

Thus we have

$$T = T_0 H(r-a) + T_1 H(a-r) - \int_{-\infty}^{\infty} \frac{(T_1-T_0)\,UaK_-(-k)}{4\pi\kappa} e^{-i(\alpha+k)x} \times \left\{\frac{K_+(\alpha)\mathscr{C}(\zeta r)}{\zeta} - \frac{U^*}{\omega}\frac{(\alpha+k)\,K_+(\alpha)\mathscr{C}(\zeta r)\,e^{-i\omega t}}{\zeta} + \frac{U^*}{\omega}\frac{(\alpha+l)\,L_+(\alpha)\mathscr{C}(\eta r)\,e^{-i\omega t}}{\eta}\right\} d\alpha. \quad (46)$$

The sound radiated directly is given by

$$c^2(\rho-\rho_0)(\boldsymbol{\xi},t) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \frac{r\tilde{\omega}\rho_0\,U^*(T_1-T_0)\,UaK_-(-k)}{4\pi\kappa T_0\,4\pi R(1+M\cos\theta)^2} \times \exp\{-i\tilde{\omega}(t-R/c) - i(\alpha+k)x - i\tilde{\omega}x\cos\theta/c - i\tilde{\omega}r\sin\theta\cos\psi/c\} \times \left\{\frac{(\alpha+k)\mathscr{C}(\zeta r)\,K_+(\alpha)}{\zeta} - \frac{(\alpha+l)\mathscr{C}(\eta r)\,L_+(\alpha)}{\eta} + \frac{\tilde{\omega}\cos\theta\,\mathscr{C}(\zeta r)\,K_+(\alpha)}{c\zeta}\right\} dr\,d\psi\,dx\,d\alpha,$$

where $M = U/c$ and $\tilde{\omega} = \omega/(1+M\cos\theta)$. Now

$$\int_0^\infty r\mathscr{C}(\zeta r) = \int_0^a \frac{rI_0(\zeta r)\,dr}{I_0'(\zeta a)} + \int_a^\infty \frac{rK_0(\zeta r)\,dr}{K_0'(\zeta a)} = 0$$

and also

$$\int_0^{2\pi} \cos\psi\,d\psi = 0$$

so at low enough frequencies we need consider only the third term when we expand the exponential $\exp(-i\tilde{\omega}r\sin\theta\cos\psi/c)$ as

$$1 - i\tilde{\omega}r\sin\theta\cos\psi/c - \tilde{\omega}^2r^2\sin^2\theta\cos^2\psi/2c^2 + \dots.$$

$$\int_0^{2\pi}\cos^2\psi\,d\psi = \pi, \quad \int_0^a \frac{r^3I_0(\zeta r)\,dr}{I_0'(\zeta a)} + \int_a^\infty \frac{r^3K_0(\zeta r)\,dr}{K_0'(\zeta a)} = \frac{2a}{\zeta^3 I_0'(\zeta a)\,K_0'(\zeta a)}$$

so

$$c^2(\rho-\rho_0)(\boldsymbol{\xi},t) \approx -\int_{-\infty}^{\infty} \frac{\tilde{\omega}\rho_0 U^*(T_1-T_0)\,UaK_-(-k)}{4\pi\kappa T_0\, 4\pi R(1+M\cos\theta)^2}\exp[-i\tilde{\omega}(t-R/c)]$$

$$\times\, 2\pi\delta(\alpha+k+\tilde{\omega}\cos\theta/c)\,\frac{2a\pi(\alpha+l)\,L_+(\alpha)}{\eta^3 I_0'(\eta a)\,K_0'(\eta a)\,\eta}\,\frac{\tilde{\omega}^2\sin^2\theta\,d\alpha}{2c^2}.$$

When $\alpha+k+\tilde{\omega}\cos\theta/c = 0$, $\eta^2 \approx -i\tilde{\omega}/\kappa$ and

$$-\frac{K_-(-k)\,L_+(\alpha)}{2I_0'(\eta a)\,K_0'(\eta a)} = \frac{K_-(-k)}{L_-(-k-\tilde{\omega}\cos\theta/c)}.$$

K_- and L_- are functions analytic in the lower half-plane that differ only in their wavenumbers $k = iU/2\kappa$ and $l = (k^2+i\omega/\kappa)^{\frac{1}{2}} \approx k(1-2i\omega\kappa/U^2)$ respectively. Consequently at low enough frequencies $K_-(-k)/L_-(-k-\tilde{\omega}\cos\theta/c) \approx 1$ and

$$c^2(\rho-\rho_0)(\boldsymbol{\xi},t) \approx -\frac{\rho_0c^2}{(1+M\cos\theta)^2}\,\frac{T_1-T_0}{T_0}\,\frac{\tilde{\omega}\kappa}{c^2}\,\frac{U^*}{c}\,\frac{\tilde{\omega}a\sin^2\theta}{c}\,\frac{\exp[-i\tilde{\omega}(t-R/c)]}{4R/a},$$

which is (18).

Appendix E

The monopole source strength is given by (34),

$$\rho_0\int_V \mathbf{u}.\mathbf{dS} = \rho_0\int_V \frac{\kappa}{c_pT}\,\frac{d^2E_V(T)}{dT^2}\left(\frac{\partial T}{\partial x_i}\right)^2 dV,$$

neglecting terms quadratic in the specific heat dE_V/dT for the vibrational mode and terms of order M^3. We further neglect those terms higher than quadratic in the temperature variations. The source strength should be evaluated in the frame of reference $\boldsymbol{\xi}$ in which the fluid at infinity is at rest, but for our purposes it is sufficient to work entirely in the frame of reference $\mathbf{x}$ in which the semi-infinite plate is stationary; the relative error will be $O(M)$. Finally, in the two-dimensional problem, gradients in the y direction are much stronger than those in the x direction, so that $(\partial T/\partial x)^2 \ll (\partial T/\partial y)^2$ (the contribution to the source strength from $(\partial T/\partial y)^2$ is larger by a factor of the Péclet number $U^2/\omega\kappa$ than the

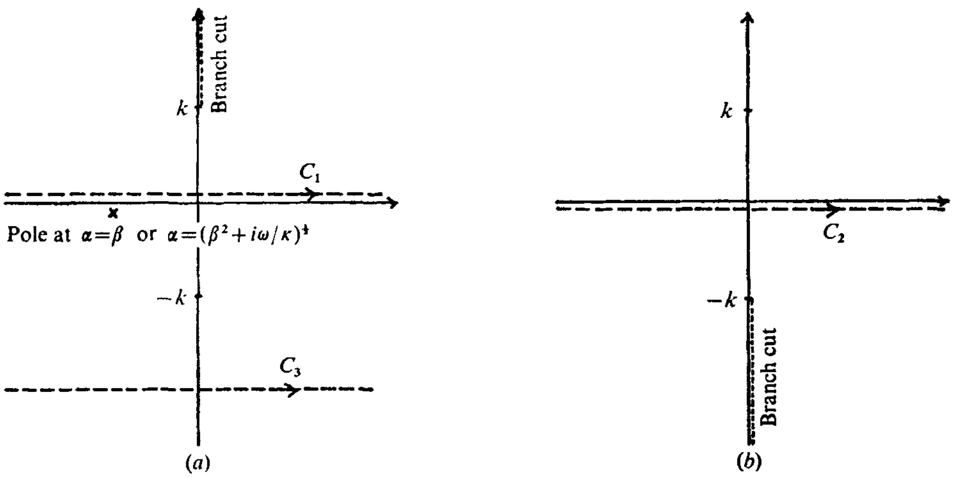

FIGURE 3. Contours for (a) the α and (b) the β integration.

contribution from $(\partial T/\partial x)^2$). Thus the dominant contribution to the monopole source strength is the fluctuating part of

$$\frac{\rho_0 \kappa}{c_p T_0}\frac{d^2E_V(T_0)}{dT^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial T}{\partial y}\right)^2 dx\,dy. \tag{47}$$

In the earlier analysis we have treated all fluctuating variables as complex; the physical quantities involved correspond to the real part of these complex variables. In the linear theory it is possible to work in terms of complex quantities throughout, but in the nonlinear theory it is important to take real parts. Thus in the expression (47) for the source strength we must substitute the real part of (43).

The fluctuating part of $(\partial T/\partial y)^2$ is given to first order in U^* by

$$2\,\mathrm{Re}\left\{\int_{-\infty}^{\infty}\frac{(T_1-T_0)\,\tilde{\zeta}\,(-2k)^{\frac{1}{2}}\exp\left[-\tilde{\zeta}|y|-i(\beta+k)\,x\right]d\beta}{4\pi i(\beta+k)\,(\beta-k)^{\frac{1}{2}}}\right\}$$

$$\times\,\mathrm{Re}\left\{\int_{-\infty}^{\infty}\frac{U^*(T_1-T_0)\,(-2k)^{\frac{1}{2}}\exp\left[-i(\alpha+k)\,x-i\omega t\right]}{4\pi i\omega}\left[\frac{-\zeta e^{-\zeta|y|}}{(\alpha-k)^{\frac{1}{2}}}+\frac{\eta e^{-\eta|y|}}{(\alpha-l)^{\frac{1}{2}}}\right]d\alpha\right\},$$

or

$$\mathrm{Re}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{U^*(T_1-T_0)^2k\tilde{\zeta}}{8\pi^2\omega(\beta+k)\,(\beta-k)^{\frac{1}{2}}}\exp\left[-\tilde{\zeta}|y|-i(\alpha+\beta+2k)\,x-i\omega t\right]\right.$$

$$\left.\times\left[-\frac{\zeta e^{-\zeta|y|}}{(\alpha-k)^{\frac{1}{2}}}+\frac{\eta e^{-\eta|y|}}{(\alpha-l)^{\frac{1}{2}}}\right]d\alpha\,d\beta\right\}+\mathrm{Re}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-\frac{iU^*(T_1-T_0)^2k\tilde{\zeta}}{8\pi^2\omega(\beta-k)\,(\beta+k)^{\frac{1}{2}}}\right.$$

$$\left.\times\exp\left[-\tilde{\zeta}|y|-i(\alpha-\beta+2k)\,x-i\omega t\right]\left[\frac{-\zeta e^{-\zeta|y|}}{(\alpha-k)^{\frac{1}{2}}}+\frac{\eta e^{-\eta|y|}}{(\alpha-l)^{\frac{1}{2}}}\right]d\alpha\,d\beta\right\},$$

where $\zeta=(\alpha^2-k^2)^{\frac{1}{2}}$, $\eta=(\alpha^2-l^2)^{\frac{1}{2}}$ and $\tilde{\zeta}=(\beta^2-k^2)^{\frac{1}{2}}$ all have positive real parts. After the integrations with respect to x and y and some further manipulations

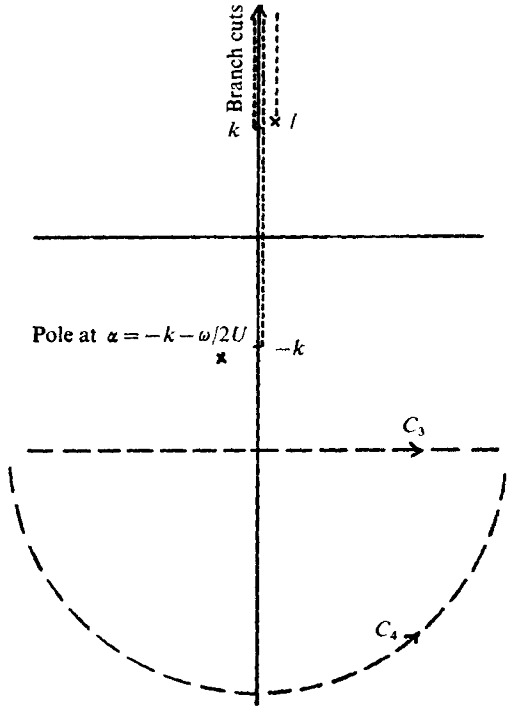

FIGURE 4. Contours for the α integration.

have been performed, the monopole source strength (47) becomes

$$\frac{\rho_0}{c_pT_0}\frac{d^2E_V(T_0)}{dT^2}\frac{U^*(T_1-T_0)^2U}{2\pi\omega}\operatorname{Re}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\delta(\alpha+\beta+2k)\,e^{-i\omega t}\right.$$
$$\times\left\{\left[\frac{-(\alpha+k)(\alpha-k)^{\frac{1}{2}}}{(\alpha+\beta)(\alpha-\beta)(\beta+k)^{\frac{1}{2}}}+\frac{(\alpha+l)(\alpha-l)^{\frac{1}{2}}}{[\alpha+(\beta^2+i\omega/\kappa)^{\frac{1}{2}}][\alpha-(\beta^2+i\omega/\kappa)^{\frac{1}{2}}](\beta+k)^{\frac{1}{2}}}\right]\right.$$
$$\left.\left.+\left[\frac{(\beta-k)^{\frac{1}{2}}(\alpha+k)^{\frac{1}{2}}}{(\alpha+\beta)(\alpha-\beta)}-\frac{(\beta-k)^{\frac{1}{2}}(\alpha-l)^{\frac{1}{2}}}{[\alpha+(\beta^2+i\omega/\kappa)^{\frac{1}{2}}][\alpha-(\beta^2+i\omega/\kappa)^{\frac{1}{2}}]}\right]\right\}d\alpha\,d\beta\right\}. \tag{48}$$

For the sake of definiteness, we assume that in the complex plane the contour C_1 for the α integration lies above the contour C_2 for the β integration (figure 3). We obtain the same final result if we assume that C_1 lies below C_2.

To evaluate the contribution from the first part of (48), we deform the contour for the α integration from C_1 to C_3, so that it passes under $\alpha=-k$. In so doing we pick up contributions from poles at $\alpha=\beta$ and at $\alpha=(\beta^2+i\omega/\kappa)^{\frac{1}{2}}$. These give us a contribution to the monopole source strength of

$$\frac{\rho_0}{c_pT_0}\frac{d^2E_V(T_0)}{dT^2}\frac{U^*(T_1-T_0)^2U}{\omega}\operatorname{Re}\left\{\int_{C_2}\left[-\delta(2\beta+2k)\,e^{-i\omega t}\frac{(\beta-k)^{\frac{1}{2}}(\beta+k)^{\frac{1}{2}}}{2\beta}\right.\right.$$
$$\left.\left.+\delta(\beta+(\beta^2+i\omega/\kappa)^{\frac{1}{2}}+2k)\,e^{-i\omega t}\frac{[(\beta^2+i\omega/\kappa)^{\frac{1}{2}}+l][(\beta^2+i\omega/\kappa)^{\frac{1}{2}}-l]^{\frac{1}{2}}}{2(\beta^2+i\omega/\kappa)^{\frac{1}{2}}(\beta+k)^{\frac{1}{2}}}\right]d\beta\right\}$$
$$=\frac{\rho}{c_pT_0}\frac{d^2E_V(T_0)}{dT^2}\frac{U^*(T_1-T_0)^2U}{\omega}\frac{\omega}{U(2\omega/\kappa)^{\frac{1}{2}}}\operatorname{Re}\{\exp(-i\omega t+\tfrac{1}{4}i\pi)\}\left\{1+O\left(\frac{\omega\kappa}{U^2}\right)\right\}. \tag{49}$$

With the contour for the α integration modified to C_3, it is possible to perform the β integration directly. When $\alpha+\beta+2k = 0$,

$$\zeta^2-\xi^2 = -4k(\alpha+k), \quad \eta^2-\xi^2 = -4k(\alpha+k+\omega/2U)$$

and branch cuts from $\beta = -k$ to $\beta = -i\infty$ run from $\alpha = -k$ to $\alpha = +i\infty$. So there is a contribution to the source strength of

$$\frac{\rho_0\kappa}{c_pT_0}\frac{d^2E_V(T_0)}{dT^2}\frac{U^*(T_1-T_0)^2}{4\pi\omega}\mathrm{Re}\left\{\int_{C_3} e^{-i\omega t}\left[\frac{(\alpha-k)^{\frac{1}{2}}}{(-\alpha-k)^{\frac{1}{2}}}-\frac{(\alpha+l)(\alpha-l)^{\frac{1}{2}}}{(\alpha+k+\omega/2U)(-\alpha-k)^{\frac{1}{2}}}\right]d\alpha\right\}.$$

Branch cuts run from $\alpha = k$, $\alpha = -k$ and $\alpha = l$ to $\alpha = i\infty$, and the pole at $\alpha+k+\omega/2U$ lies above the contour, so we can further deform the contour for the α integration to C_4 (see figure 4), the semicircle at infinity in the lower half-plane, without crossing any poles.

$$\left[\frac{(\alpha-k)^{\frac{1}{2}}}{(-\alpha-k)^{\frac{1}{2}}}-\frac{(\alpha+l)(\alpha-l)^{\frac{1}{2}}}{(\alpha+k+\omega/2U)(-\alpha-k)^{\frac{1}{2}}}\right] \sim \frac{-\omega^2\kappa}{2U^3\alpha} \quad \text{as} \quad |\alpha|\to\infty$$

so the contribution to the source strength from this integral is

$$\frac{\rho_0\kappa}{c_pT_0}\frac{d^2E_V(T_0)}{dT^2}\frac{U^*(T_1-T_0)^2}{4\pi\omega}\frac{\pi\omega^2\kappa}{2U^3\alpha}\mathrm{Re}\{-i\,e^{-i\omega t}\},$$

smaller than (49) by $O((\omega\kappa/U^2)^{\frac{1}{2}})$.

The second part of (48) is evaluated by first deforming the contour for the β integration from C_2 to C_3, since here branch cuts for the β integration are in the upper half-plane. This is achieved without crossing any poles. The α integration is then simply performed. We can further deform the contour for the β integration to C_4, the semicircle at infinity in the lower half-plane, and the contribution to the source strength is again smaller than (49) by $O((\omega\kappa/U^2)^{\frac{1}{2}})$.

The expression for the monopole source strength is therefore given by (49),

$$\frac{\rho_0c^2}{2^{\frac{1}{2}}\omega}\frac{(T_1-T_0)^2}{T_0^2}\frac{T_0}{c_p}\frac{d^2E_V(T_0)}{dT^2}\frac{U^*}{c}\left(\frac{\omega\kappa}{c^2}\right)^{\frac{1}{2}}\mathrm{Re}\{\exp(-i\omega t+\tfrac{1}{4}i\pi)\},$$

neglecting terms smaller by factors of M, the Mach number, of $\omega\kappa/U^2$, the reciprocal of the Péclet number, of $R^{-1}dE_V/dT$, the relative specific heat for the vibrational mode, or of $(T_1-T_0)/T_0$, the relative temperature difference. The radiated far-field pressure is given by the real part of

$$p = \rho_0c^2\frac{(T_1-T_0)^2}{T_0^2}\frac{T_0}{c_p}\frac{d^2E_V(T_0)}{dT^2}\frac{U^*}{c}\left(\frac{\omega\kappa}{c^2}\right)^{\frac{1}{2}}\frac{\exp[-i\omega(t-R/c)]}{(16\pi\omega R/c)^{\frac{1}{2}}},$$

which is (37).

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